Topics in Spatial Mathematics: Area

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Introduction

In a recent book, *Spatial Mathematics: Theory and Practice through Mapping* (Arlinghaus, S. L. and

Kerski, J., CRC Press, 2013), we emphasized theory ahead of example. Theory was drawn from far-flung

parts of pure mathematics and applied, in mapping contexts, to a wide range of real-world issues. The

intent was to demonstrate the power of method and to offer readers of varying backgrounds and

interests, alternate ways to consider the use of pure mathematics in mapping contexts. Some of these

contexts were traditional, and some were not. Detailed, worked examples were available in the eBook

version.

Conceptual Rationale

Here, we tighten the focus to offer selected mathematical tools, that environmental scientists or

engineers might normally acquire in the course of standard undergraduate or graduate curricula, and

illustrate how to select the spatial components and apply them to real-world environmental problems.

The selections made are 'standards-based' on typical curricular topics.   The one offered in this article

has been classroom-tested over a number of years by W. C. Arlinghaus at Lawrence Technological

University, Department of Mathematics.

Often, environmental or other real-world application comes after the mathematics, itself. Here, we use

the environmental problem as the driving force. Thus, in looking for 'standards' we turn, in part, to the

most recent set enunciated by the National Geography Standards Index of the National Geographic

Society; while these are a foundation intended for the use of pre-collegiate teachers, we believe that what one learns well in a pre-collegiate setting, should (in the best of all possible worlds), translate to being extended into a more advanced formal educational setting.   These Standards thus form one set that applies widely, as a foundation for geographical and environmental curricula at all academic levels. Their creators note (http://education.nationalgeographic.com/standards/national-geography-standards/) that:

> *Understanding and responding to the challenges and opportunities of the world in the twenty-first century will require many skills; the capacities to think and communicate mathematically and scientifically will remain at a premium. Geographic literacy will also be necessary for reasons of enhancing economic competitiveness, preserving quality of life, sustaining the environment, and ensuring national security. As individuals and as members of society, humans face decisions on where to live, what to build where, how and where to travel, how to conserve energy, how to wisely manage scarce resources, and how to cooperate or compete with others.*

The current standards (2012, National Geography Standards, 2nd Edition) enumerates 18 *Standards* within 6 *Essential Elements*. In this particular application, and in subsequent ones to come, we examine a selection of concepts from that set, and extend them to focus on different individual 'Topics'. The meaning and use of fundamental spatial concepts such as partition, area, shape, location, distance, direction, scale, connection, and hierarchy are critical.

Teaching Rationale

Anyone who has taught students to learn mathematical tools knows that a few students easily grasp equations and mathematical notation while many others struggle. The wisdom of the Mathematical Association of America in introducing its series in the *Mathematical Magazine*, called 'Proofs Without Words' underscores the importance of this distinction. In the words of Lynn Arthur Steen, co-editor of

that journal with J. Arthur Seebach in 1976 (when the original "Proofs Without Words" column premiered):

> *As a teacher, I often urged students to remember geometric diagrams that encapsulate the essence of key mathematical relationships or theorems. For most people, visual memory is more powerful than linear memory of steps in a proof. Moreover, the various relationships embedded on [sic] a good diagram represent real mathematics awaiting recognition and verbalization. So as a device to help students learn and remember mathematics, proofs without words are often more accurate than (mis-remembered) proofs with words.*

It is a series with numerous spin-offs, including, for example, Roger B. Nelsen's (1993) fine work, *Proofs Without Words: Exercises in Visual Thinking*. One of the most famous examples, widely cited in a variety of works, is President Garfield's Proof of the Pythagorean Theorem. We repeat it here, simply to pique the reader's curiosity!

The focus in Garfield's proof (Figure 1) is on the formula for the area of a trapezoid as half sum of the bases times the altitude, or as *((a + b)/2)*(a + b)*. (See for example, Pappas, 1989.) The angles *(ABC), (BCD),* and *(DEA)* are right angles. This value can also be computed as the sum of areas of the three right triangles: *ab/2 + ab/2 + cc/2*. Simplifications yields $a^2 + b^2 = c^2$
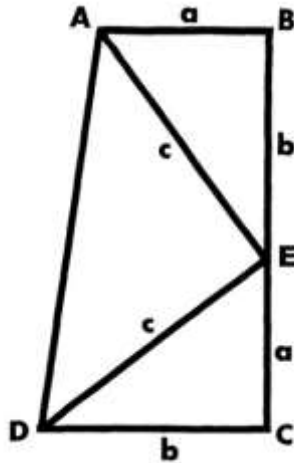
Figure 1. President Garfield's proof of the Pythagorean Theorem

Topic: Area

All too often, the concept of area is taken for granted. How many square miles is the state of Iowa?

"Just look it up in a printed reference work or online," is a common answer. Looking online, however,

reveals a number of different answers. Wikipedia shows the area to be 56,272 square miles. Netstate,

has it at 56,276 square miles. In addition, Netstate notes that the land area is 55,875 square miles and

the water area 401 square miles. The two do add to the stated total, and that at least is reassuring. The

Iowa Quick Facts State Data Center shows a value of 55,857.1 square miles of land area while the Iowa

Quick Facts from the US Census Bureau has the land area at 55,857.13 square miles. The USGS, on the

other hand shows a total land area of 55,857 square miles and a water area of 416 square miles. The

total area they give is 56,273 square miles which once again is the sum of the parts, offering some

reassurance to the researcher.

It appears that some of the differences in values are due to round off or truncation of decimal values.

Others might be typographical errors. Yet, one cannot assume such. Differences in scale of

measurement, difference in position of control points as benchmarks for survey, difference in map

projection, and a host of other issues might cause these variations.   To see how map projection, for

instance, can affect the visualization of area, consider the different shapes on four different map

projections, all in common use (Figure 2). Each ellipse surrounds an area of 1000 km^2 in or near a

location in southern Greenland (Esri, 2015).
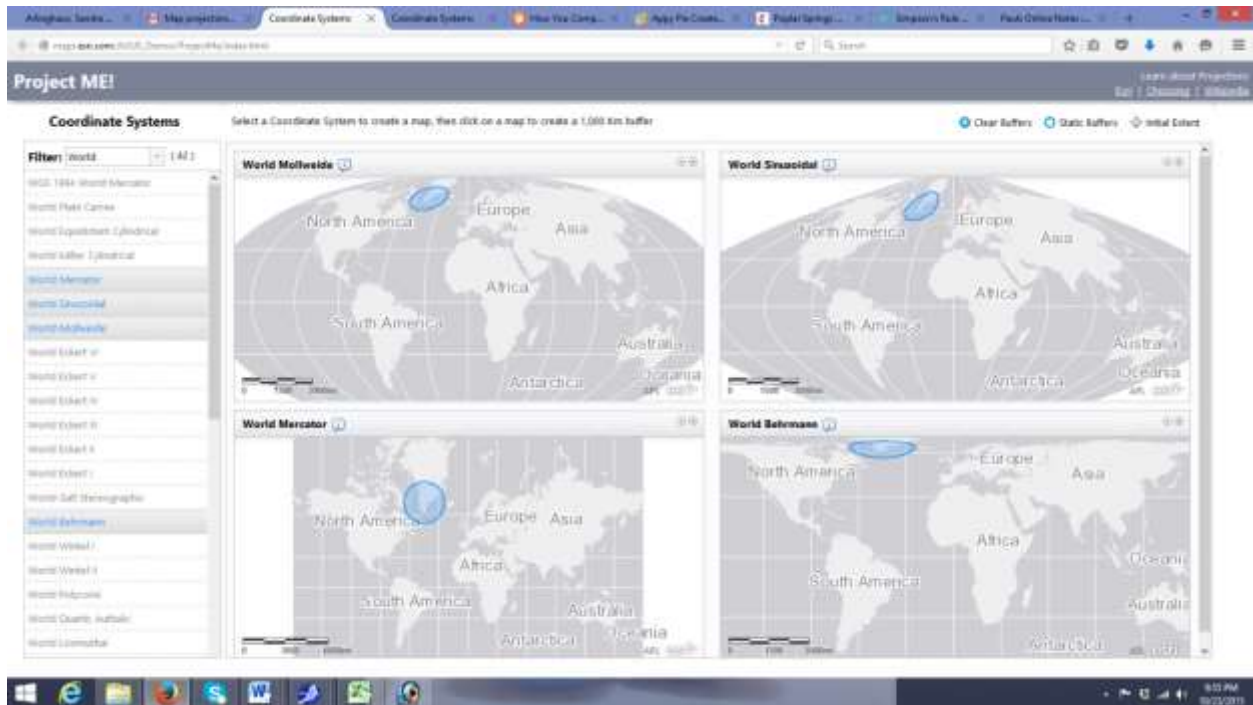
Figure 2. Blue elliptical area measures 1000km^2. The variation in appearance is due to the manner in which a globe, representing the surface of the Earth, is projected into a plane. (Esri, 2015.)

Still other questions might center on whether to count the water area as part of the total area. There is, after all, land under the water. On the other hand, though, the water is on the surface and if it is surface area that is of interest, then perhaps the water area should not count. Then again, some streams are intermittent (dry part of the time) and so perhaps one should count intermittent streams as part of the total area but not include perennial (always wet) streams. Clearly the environmental situation drives the choices made in what data sets to include.

Environmental Importance of Area

To measure the area of a circle or rectangle is trivial; we all learn to measure the area of certain regular shapes early in life. Then, we learn to partition more complex shapes into a sum of shapes we already know how to measure. President Garfield's proof of the Pythagorean Theorem drew on that idea: he saw the trapezoid as composed of a rectangle and triangles. Most real-world shapes, however, are not regular. There are simple ways to measure irregular shapes that can be mastered with little formal education. Others, that are more accurate, require more advanced education. Electronic tools that will measure area are all based on one mathematical tool or another. As is the case with printed/online estimates of area, so too it is the case with contemporary electronic tools, the mathematics is the driving force for calculation. Thus, to have a clear grasp of the meaning of measurements, one must also know how and why they work.

How do we measure the surface area of an oil spill on the ocean, for example? How accurate does such a measurement need to be? Surely the shape of the spill will not be any regular shape that we do know how to measure. The level of accuracy needed will depend on what the goal is to do with the spill. If it is a matter of simply showing a television audience that the spill covers a significant amount of ocean surface, then accuracy is probably not much of an issue. Here, a simple technique, such as covering an image of the spill with a sheet of graph paper, and counting the squares as 'in' or 'out' of the spill will

give a reasonable size of the spill (given that the side of a square of the graph paper has been calibrated with some measuring stick). What to do with boundary square tiles is always an issue with this sort of approach (Nystuen, 1967). Do they count as 'in' if more than half the square lies 'inside'? Do we look, instead, at balancing 'overfit' and 'underfit' of the squares with respect to the entire shape? Either the local or global (with respect to the entire shape) measure might offer insight, as might a mix of techniques. Readers interested in the advanced mathematical issues here might wish to read a classical work, such as that of Halmos (1950).

If, however, greater accuracy of the area of the spill is needed, as it might be if a budget for remediation of the spill is based on so many dollars per square foot or square mile of spill, then it is critical to have the estimate of the area large enough so that remediation of the whole spill will take place, yet small enough to optimize expenditure of precious funds. Planning environmental restoration often involves many disparate variables which may, as in this case, hinge on having a clear picture of area.

Beyond Graph Paper

An approach to measuring irregular shapes, that goes a bit beyond graph paper, and offers a systematic means for replication of results involves the so-called 'midpoint rule' often taught in elementary calculus classes. It is a systematic extension of the graph paper idea.

Consider a function *y = f(x)* defined over the closed interval (endpoints included) [*a, b*] (Dawkins, 2015). Divide this interval into *n* subintervals of equal width, *h.* Thus,

$$h = \frac{b - a}{n}$$

Denote each of the intervals

$$[x_0, x_1], [x_1, x_2], \dots , [x_{n-1}, x_n] \; where \; x_0 = a \; and \; x_n = b$$

For each subinterval let x_i^* be the midpoint of the subinterval. Figure 3 shows a graph of $f(x)$ with $n = 6$ and with rectangles sketched in for each of the six subintervals with a height of $f(x_i^*)$ or y_i*.
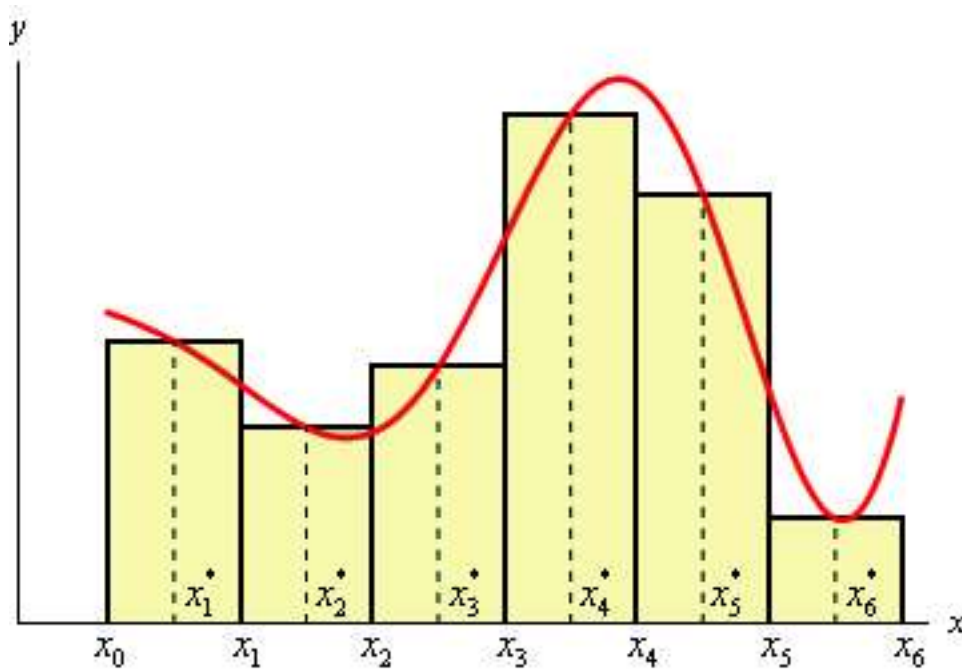
Figure 3. Approximation to the area under a curve using the 'midpoint' rule. Derived from a figure of Dawkins, 2015.

Note the overfit and underfit of the rectangles; that is the pattern that lets one see this process as an extension of the graph paper approximation. Use of the subinterval midpoints as the height of the rectangle is what makes this process, unlike the graph paper process, one that is systematically replicable: the amount of overfit and underfit is no longer a subjective judgment call: it is uniquely determined by the height of the function at the midpoint of a subinterval.

The approximation to the area, *A(x)*, under this curve is thus captured as a sum of the areas of

six rectangles: *A(x)* ≈ *h*y₁** *+ h* y₂** *+ h* y₃** *+ h* y₄** *+ h*y₅** *+ h*y₆**

While one does not need calculus to understand the idea, or to execute the method, it is easy

to express the concept using calculus and such expression will lead eventually to other ways to

capture area under a curve as one assumes that increasing the fineness of the partition will

increase the accuracy of the approximation. Thus, generally, factoring out an *h* term,

We can easily find the area for each of these rectangles and so for a general *n* we get the

midpoint rule that, expressing the area *A(x)* as a definite integral,

$$\int_a^b f(x)dx \approx h[y_1^* + y_2^* + \cdots + y_n^*]$$

Trapezoidal Rule

Reflecting on President Garfield's proof of the Pythagorean Theorem (from the Introduction) using a

trapezoid to simplify matters, one might be drawn naturally to adjust the midpoint rule (above) by tilting

the tops of the rectangles to form trapezoids. Indeed, exactly the same sort of approach works (with

subinterval width denoted as *h*), as Figure 4 suggests.   A finer partition will be likely to improve the

approximation. One might also be tempted to think that the use of trapezoids will always yield a better

approximation than the use of rectangles. That is not the case however; it depends on how well overfit

and underfit balance (the subjective concept often invoked in the field using graph paper). Note the

considerable overfit of the last trapezoid on the right; contrast that with the pattern in Figure 3. The

choice for *n*, the number of subintervals, may determine which Rule gives a better fit in any particular

instance.
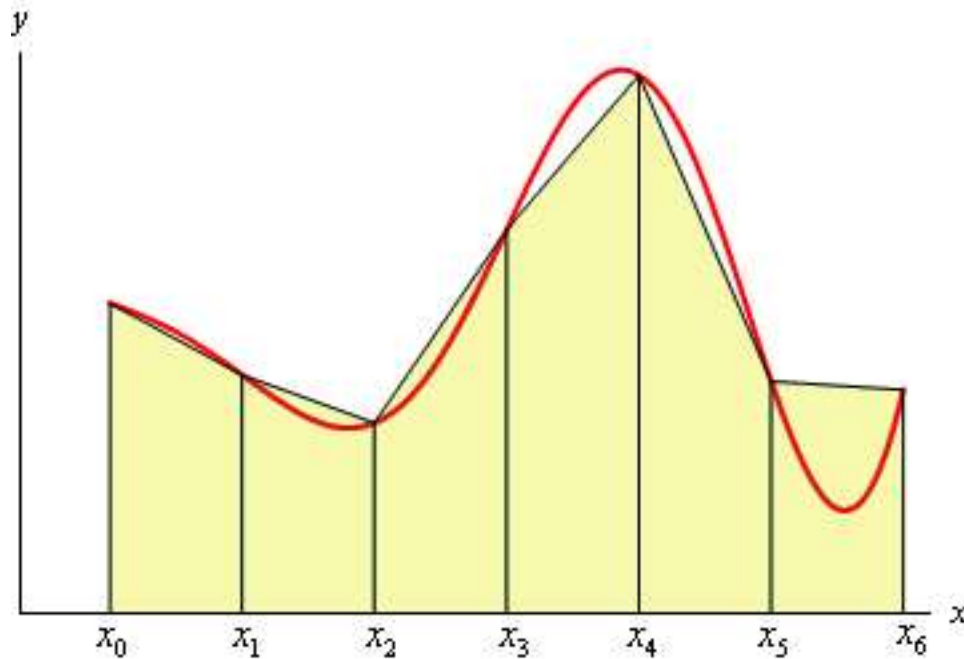
Figure 4. Trapezoids used in place of the rectangles of Figure 3. Derived from a figure of Dawkins, 2015.

Here, the notational characterization that parallels that of the midpoint rule is:

$$\int_a^b f(x)dx \approx \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n)$$

After simplification involving factoring out h/2 and gathering like terms:

$$\int_a^b f(x)dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n]$$

The Fundamental Theorem of Calculus: Geometric Meaning

In the 'midpoint' rule and the 'trapezoidal' rule, we used simple arithmetic and geometry, coupled with

systematic use of ideas of 'overfit' and 'underfit' to approximate areas under curves. The procedure

derived from looking at simple cases was generalized and characterized notationally as a procedure to

approximate a definite integral representing area under a curve between upper and lower values. These ideas can be carried forward so that any *f(x)* that has an antiderivative can be systematically treated to find the area under the curve representing that function (Wikipedia, 2015). In the cases above, we did not know the equation for f(x) and certainly did not know about any antiderivative. Had we known such, exact evaluation of the area would have been possible using the Fundamental Theorem of Calculus. The Fundamental Theorem of Calculus is important, indeed 'fundamental', because it links the concept of the derivative of a function with the concept of the integral of the function: 'derivatives' and 'integrals' are root concepts.

Often this fundamental theorem is expressed in two parts. In the first part, the definite integration of a function is related to the antiderivative of the function, and can be reversed by differentiation (process of finding derivatives). It guarantees the existence of antiderivatives for continuous functions. The second part of this theorem states that the definite integral of a function can be computed from any one of its infinite number of antiderivatives. In practical application this theorem simplifies the computation of definite integrals and permits exact determination of areas under curves with known antiderivatives. To see, visually, how the linkage between derivatives and integrals arises, consider the schematic in Figure 5.
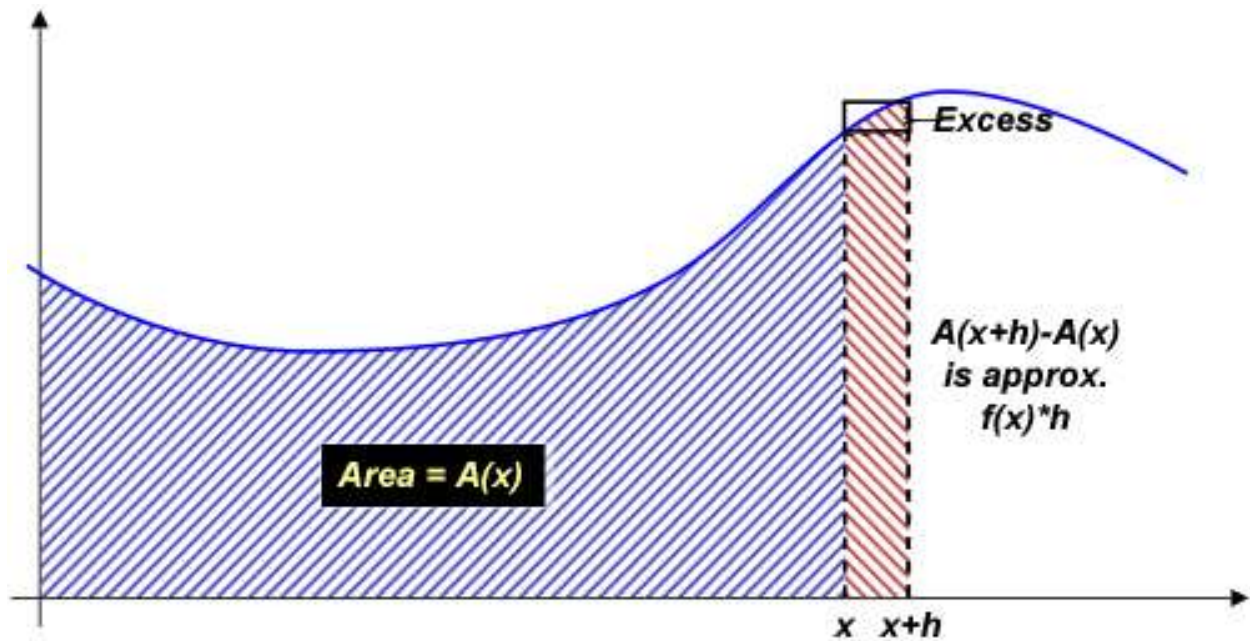
Figure 5. Underfit/overfit concepts as 'Excess' used in the Fundamental Theorem of Calculus. Figure derived from one in Wikipedia, 2015.

The area shaded in red stripes can be estimated as h times $f(x)$. Alternatively, if the function $A(x)$ were known, it could be computed exactly as $A(x + h) - A(x)$. These two values are approximately equal, particularly for small h.

For a continuous function $y = f(x)$ whose graph is plotted as a curve, each value of x has a corresponding area function $A(x)$, representing the area beneath the curve between 0 and x. The function $A(x)$ may not be known, but it is given that it represents the area under the curve.

The area under the curve between x and $x + h$ could be computed by finding the area between 0 and $x + h$, then subtracting the area between 0 and x. In other words, the area of this "sliver" would be $A(x + h) - A(x)$.

There is another way to *estimate* the area of this same sliver. As shown in Figure 5, *h* is multiplied by *f(x)* to find the area of a rectangle that is approximately the same size as this sliver. So:

$$A(x + h) - A(x) \approx f(x)h$$

In fact, this estimate becomes a perfect equality if we add the red portion of the "excess" area (compare to ideas of overfit and underfit) shown in the diagram. So:

$$A(x + h) - A(x) = f(x)h + (Red\ Excess)$$

Rearranging terms:

$$f(x) = \frac{A(x + h) - A(x)}{h} - \frac{Red\ Excess}{h}$$

As *h* approaches 0 in the limit, the last fraction can be shown to go to zero. This is true because the area of the red portion of excess region is less than or equal to the area of the tiny black-bordered rectangle; the area of that tiny rectangle, divided by *h*, is simply the height of the tiny rectangle, which can be seen to go to zero as *h* goes to zero.

Removing the last fraction gives:

$$f(x) = \lim_{h \to 0} \frac{A(x + h) - A(x)}{h}$$

Note that this expression is exactly that given in one definition of derivative. It can thus be shown that *f(x)* = *A′(x)*. That is, the derivative of the area function *A(x)* is the original function *f(x)*; or, the area function is simply an antiderivative of the original function. Computing the derivative of a function and "finding the area" under its curve are "opposite" operations. This idea is the basis of the Fundamental Theorem of Calculus.
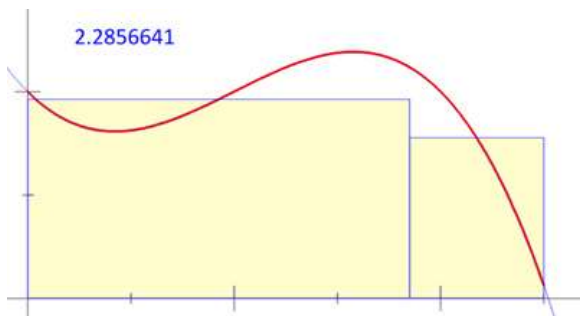
Spatial View

Visually, imagine using rectangles of thinner and thinner breadth to estimate successive approximations to area (Figure 6, a-f, Wikipedia 2015). The link shows an animation of five fluctuating approximations to area, *A(x)*, under a curve based on these figures.
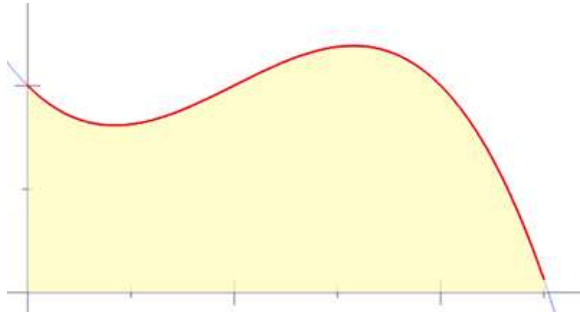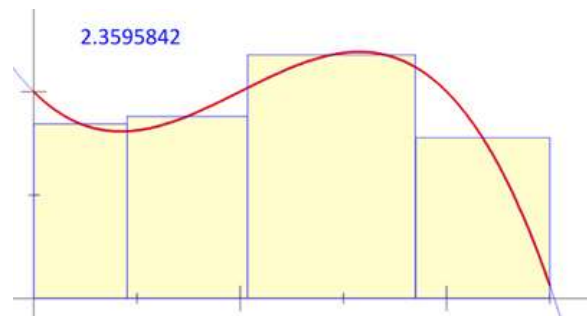

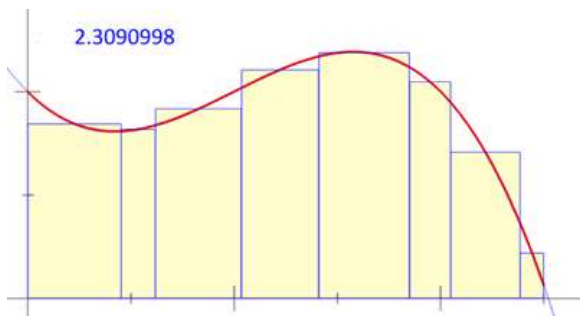**Figure 6a.** Desired area shaded.
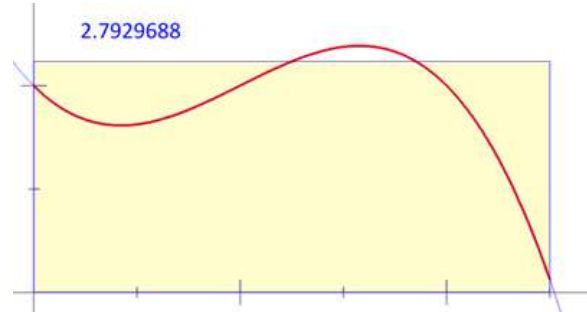
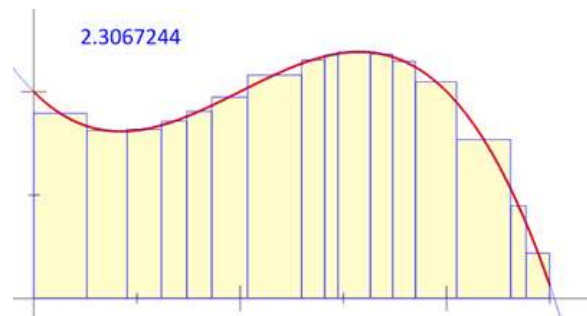**Figure 6b.** First approximation.

Figure 6c. Second approximation.

Figure 6d. Third approximation.

Figure 6e. Fourth approximation.

Figure 6f. Fifth approximation.

The Fundamental Theorem of Integral Calculus assures us that as process continues, the approximate areas will converge to the actual area. When we know the antiderivative of the integrand, and can write it down and evaluate it from one limit of the integral to the other, then we can find area between the limits (although executing the task is sometimes difficult). Generally, if a function that has an antiderivative describes a curve, we can find the area between it and an axis. Regions bounded by multiple such curves can be partitioned and areas of subregions found and added together. The concept of partition is a powerful spatial concept, alone, and its power is easily visualized when compounded with the concept of area. More of that power will be developed in a separate Topic on Partition.

Simpson's Rule

The Fundamental Theorem of Calculus assures us that the area under a curve can be found exactly when the antiderivative of the associated function can be determined. In practical situations, however, the researcher may have no knowledge of an equation derived from empirical data, let alone its antiderivative. Clearly, either the midpoint rule or the trapezoidal rule might be used to estimate areas, as techniques of numerical integration, in such situations. But, when the curve is highly curvy, the amount of overfit and underfit may be large and either of those rules may yield bad approximations. Simpson's Rule offers a system, with some abstract similarity to the midpoint and trapezoidal rules for numerical integration, that employs pieces of parabolas, rather than line segments, to approximate the edge of the sought-after area.

As with the two previous rules, the interval over which the area is calculated will be divided into *n* subintervals (Figure 7). Again, the width of each subinterval is,

$$h = \frac{b - a}{n}$$

However unlike the two previous rules, *n* must be an even number (divisible by 2). Because it takes three points to determine a parabola, the quadratic equation associated with a parabola piece must agree with three successive points from the subintervals, covering two areas (see the color coding in Figure 7). Each of the approximations is colored differently, covering two adjacent subintervals, thereby highlighting the necessity to have *n* even (Dawkins,, 2015).
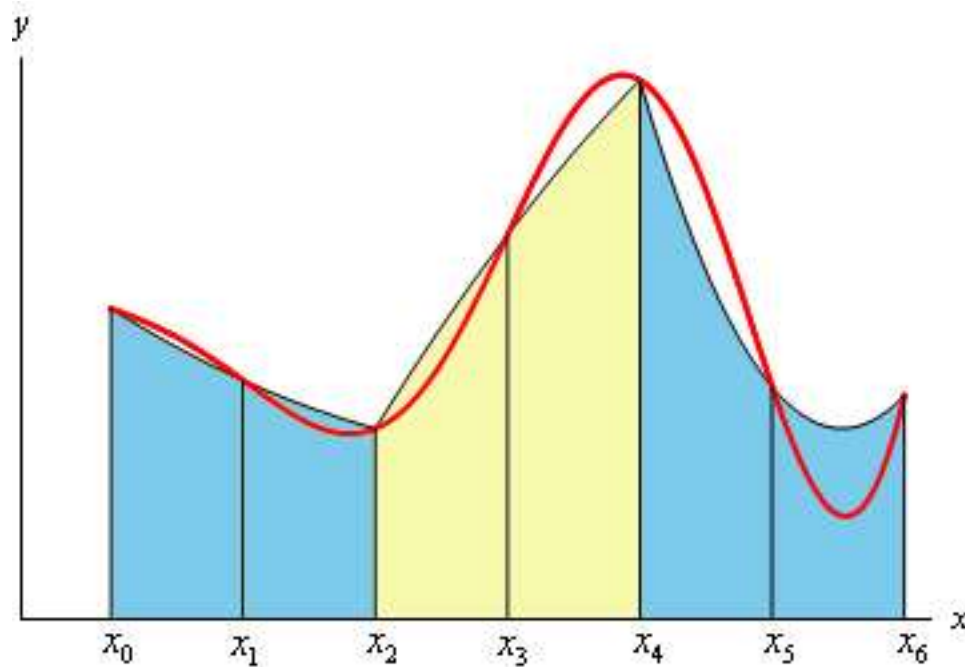
Figure 7. Parabola pieces pass through three points of the partition, forcing *n* to be even. It takes three points to uniquely determine a parabola. Derived from a figure of Dawkins, 2015.

As before, some of the approximations do a better job of fitting the red curve than do others. Some have sharper curvature than others, but all are parabola pieces. The notation used to represent this method will be based on quadratic functions: remember that there is a one-to-one correspondence between parabolas and quadratic equations. Since it uses quadratic polynomials to approximate functions, Simpson's rule actually gives exact results when approximating integrals of polynomials up to cubic degree (Swokowski, 1979).

First Step in Deriving Simpson's Rule

To derive the formulation for Simpson's rule, we show first that the area under any approximation to a general parabola, from –h to h is A(x) = h/3(y₀ +4y₁ +y₂). Figure 8a shows the set up with the subintervals situation symmetrically on either side of the origin. To illustrate the notation, we use portions of a general quadratic, cx^2 +dx +e = 0. If c is not zero, then the graph of this function is a parabola. Thus, the area *A(x)* under this general quadratic is:

$$\int_{-h}^{h} (cx^2 + dx + e)dx = \frac{cx^3}{3} + \frac{dx^2}{2} + ex \ ]_{-h}^{h} = \frac{h}{3}(2ch^2 + 6e)$$
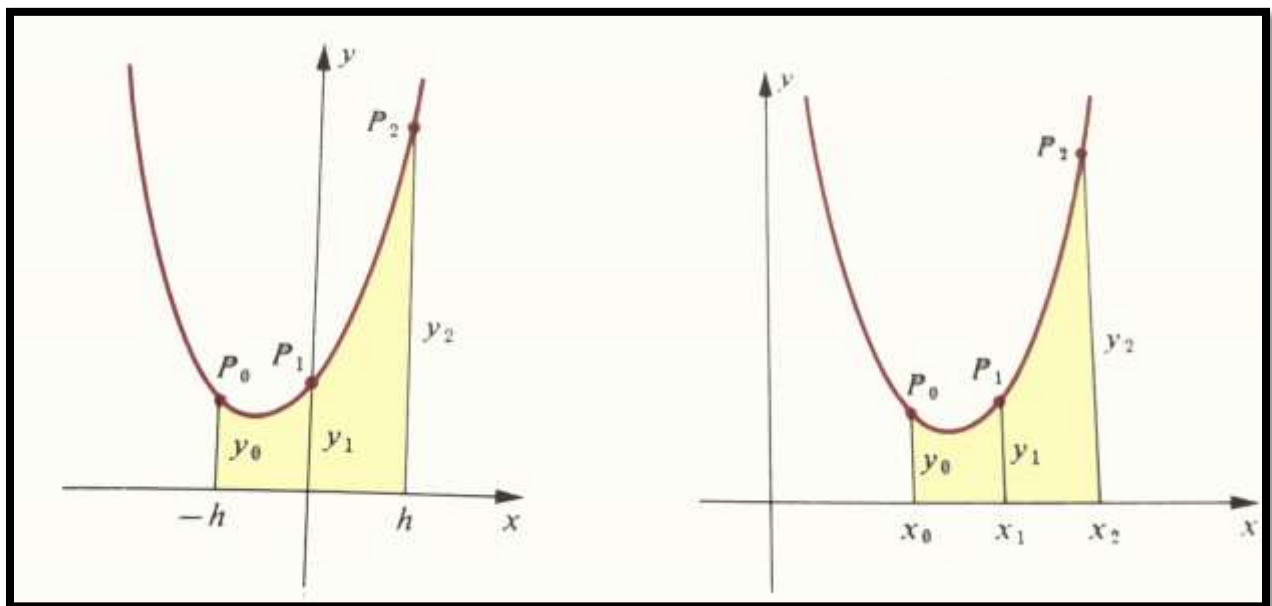

Figure 8. a) Subintervals situated symmetrically around the origin. b) Curve and associated subintervals translated to an arbitrary position. Area under the curve is the same in both figures. Figure derived from one in Swokowski, 1979.

Since the coordinates of P_0, P_1, and P_2 satisfy the equation $y = cx^2 + dx +e$, it follows that

$$y_0 = ch^2 - dh + e$$
$$y_1 = e$$
$$y_2 = ch^2 + dh + e$$

from which it follows that $y_0 + 4y_1 + y_2 = 3ch^2 + 6c$.

Thus, $A(x) = (h/3)*(y_0 + 4y_1 + y_2)$ as desired.

If the points P_0, P_1, and P_2 are translated horizontally, as in Figure 8b, the area under the curve remains

the same. Thus, $A(x) = (h/3)*(y_0 + 4y_1 + y_2)$ is true for any points P_0, P_1, and P_2, so long as they determine

adjacent intervals as in the Figure.

Second Step in Deriving Simpson's Rule

Figure 9 suggests a general situation with an even number of intervals, as required.
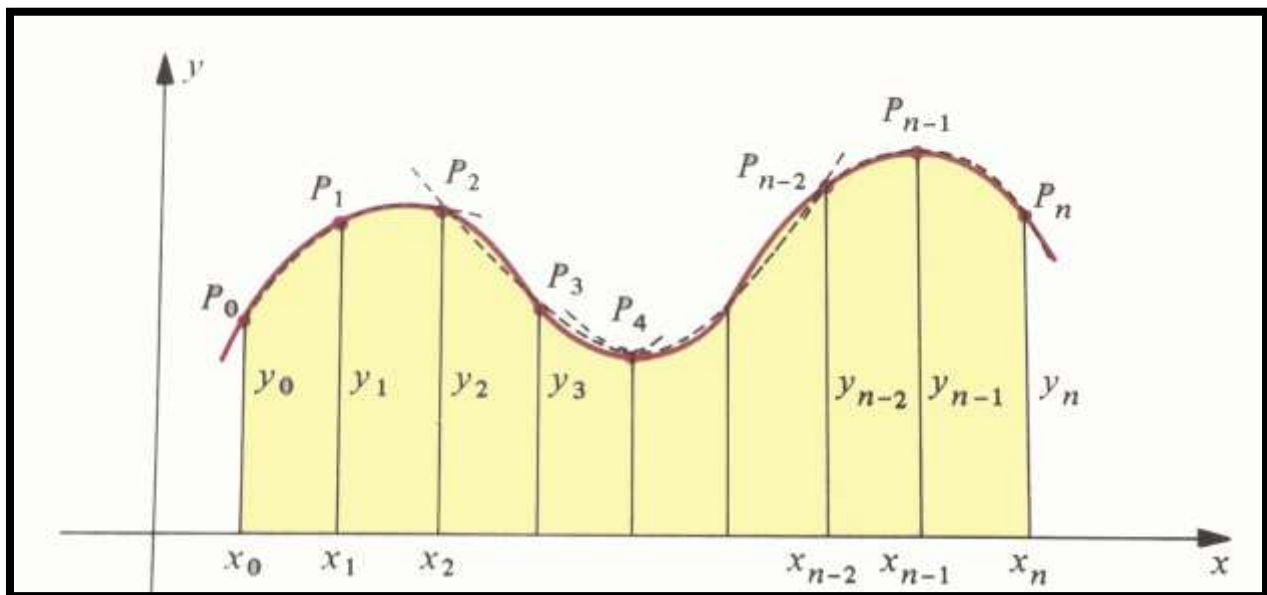
Figure 9. Generalizing Simpson's Rule. Figure derived from one in Swokowski, 1979.

The general approximation on the intervals from $[x_{i-1}, x_i]$ and $[x_i, x_{i+1}]$ is

$$A_i = \frac{h}{3}(y_{i-1} + 4y_1 + y_{i+1})$$

If we use *n* subintervals the integral is then approximately,

$$\int_a^b f(x)dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Factoring out the h/3 and gathering like terms yields the general Simpson's Rule.

$$\int_a^b f(x)dx \approx \frac{h}{3}(y_0 + 4y_1 + 2y_2 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

In this case notice that all the function evaluations at points with odd subscripts are multiplied by 4 and all the function evaluations at points with even subscripts (except for the first and last) are multiplied by 2. Remember this pattern to make the rule fairly easy rule to remember.

Coming Full Circle

At the beginning of this Topic, we discussed discrepancies in the measurement of the state of Iowa that appear in various sources. If one wished to check for himself or herself what that area might be, the tools are available to do so. A GPS might be used by an individual to measure small areas; it hardly seems reasonable, however, to consider doing such for an area as large as the state of Iowa. The shape of the state is fairly close to rectangular, so that one might use a simple rectangle as a crude approximation. The northern boundary is a straight line, along a small circle of latitude. The southern boundary is also measured, except for a small triangle at the southeastern corner, along a small circle of latitude; thus, for the most part, these boundaries are parallel (as the planes of small circles are all parallel to the equatorial plane, and hence to each other). The eastern and western boundaries of Iowa follow major river courses and are quite curvy, as Figure 10 demonstrates.

In the worked application below, we show how to use Simpson's Rule to calculate the area of Iowa.

The following facts are needed, in addition to Simpson's Rule, to help compute the area:

a) The circumference of the earth is approximately 24,901 miles.
b) We need to use some software to find latitude and longitude of selected points.
c) We need a partition into an even number of subintervals in the manner of Simpson's Rule.
d) The circumference of the circle at latitude α is $24901\cos\alpha$. This circle is divided into 360 degrees of longitude.
e) The distance between points of equal longitude is measured along great circles, which have the same circumference as the equator.
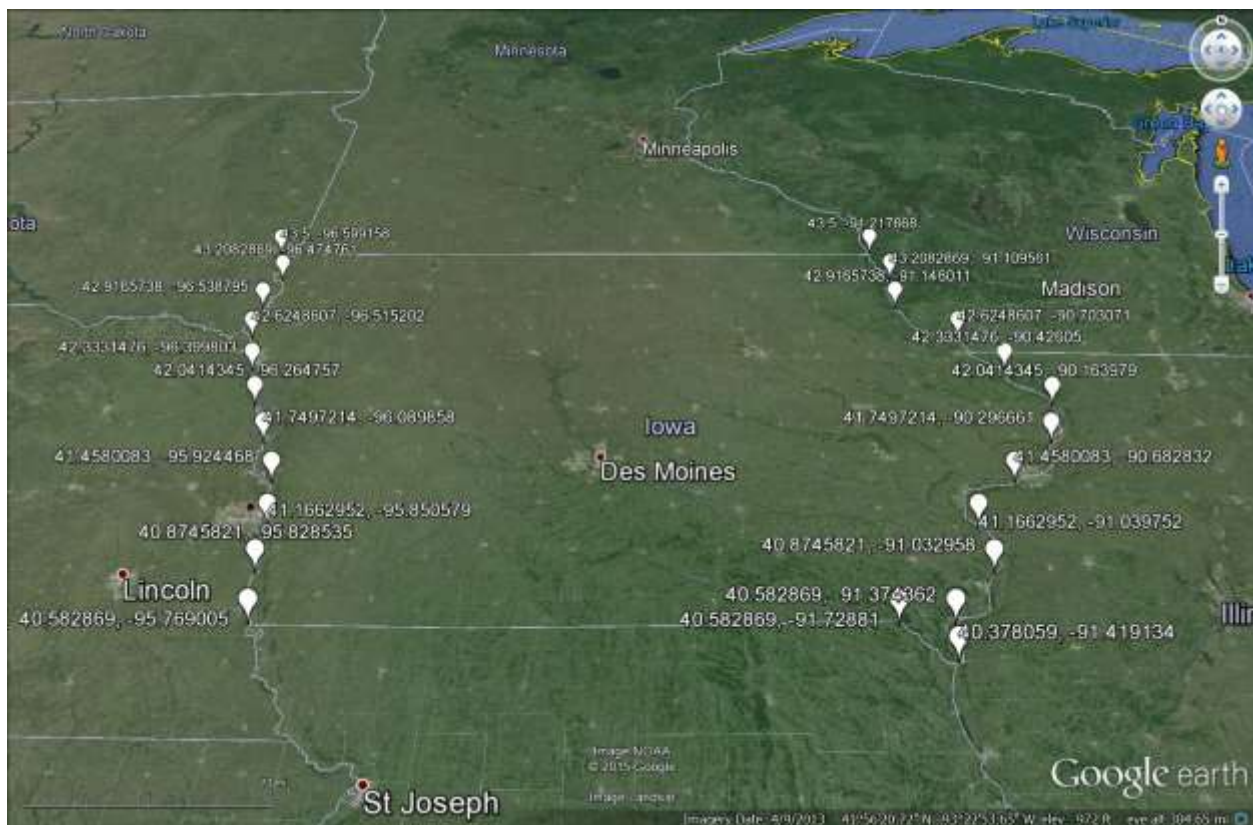
Figure 10. The state of Iowa.

To address points a) through e) above, we locate, using coordinates from Google Earth free software, a set of ten locations along the curvy western and curvy eastern boundary of Iowa. These points will be used with Simpson's Rule to calculate the area of the state. The area of the small triangle at the southeast corner of the state will be calculated separately; the problem of finding the area partitions the

process. These ten points on the east and west borders will be spaced evenly at a spacing of *h*. Table 1

shows the latitude and longitude for each of these 10 points.

Table 1. Coordinates for use in Simpson's Rule.

Latitude	E Iowa	W Iowa	E-W length
43.5	-91.217668	-96.599158	270.0095
43.2082869	-91.109561	-96.474761	270.4893
42.9165738	-91.146011	-96.538795	273.1767
42.6248607	-90.703071	-96.515202	295.809
42.3331476	-90.42605	-96.399803	305.4555
42.0414345	-90.163979	-96.264757	313.3935
41.7497214	-90.296661	-96.089858	298.9556
41.4580083	-90.682832	-95.924468	271.7182
41.1662952	-91.039752	-95.850579	250.5042
40.8745821	-91.032958	-95.828535	250.8185
40.582869	-91.72881	-95.769005	212.2388

Table 1 also calculates the lengths of E-W segments in miles. For example, the length of the line

segment at the northern boundary between the west boundary and the east boundary is (-

91.2177+96.5992)24901cos43.5° / 360 = 270.0095 sq. miles.

To find the value for *h*, simply subtract adjacent entries in the latitude column and divide by 360, so that

here *h* is (43.5 - 43.20829)24901 / 360 = 20.17763306.

Then, with all these values, Simpson's Rule, using the calculated value of *h* together with the calculated

E-W segments as the y_i values, gives the value of:

A= (h/3)*(y$_0$ + 4y$_1$ + 2y$_2$ + 4y$_3$ +2y$_5$ + 4y$_6$ + 2y$_7$ + 4y$_8$ + 2y$_9$ + y$_{10}$)

= 20.17763306/3(270.0095 +4*270.4893+2*273.1767 + 4*295.809 + 2*305.4555 + 4*313.3935 + 2*298.9556 + 4*271.7182 +2*250.5042 + 4*250.8185 + 212.2388)

= 56143.22883 square miles, for the area of Iowa less the triangle at the southeast corner.

The calculations were made in an Excel spreadsheet, using more decimal values than are visible in the text.

Table 2 shows the dimensions for the triangle calculations (again made in an Excel spreadsheet):

Table 2. Calculations of triangle at southeastern corner of Iowa.

South Boundary Triangle Base:		-91.374362-(-91.72881))*24901/360 = 24.51697124
South Tip, Triangle Height:		(40.582869-40.378059)*24901/360=14.16659392
Area of Triangle (0.5 bh):		173.6609878

Thus, the calculated area of Iowa is the area calculated using Simpson's Rule plus the area of the triangle: 56143.22883 + 173.6609878 = 56316.88982 square miles. Using one published value of 56,363.3 square miles, as an 'actual' measure of accepted area, enables us to calculate the percentage error: ((actual – estimated)/actual)*100 = 0.08234113. The Simpson's Rule estimate is not bad! And, it is far easier to do today, using a spreadsheet and remotely sensed coordinates, than it once was!

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