

# On Mathematics of Navigational Calculations for Meridian Sailing

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**Abstract.** Meridian sailing is actually a special case of sailing uniquely determined by relative coordinates  $\Delta\varphi, \Delta\lambda$ . In this note we discuss explicit finding of certain navigational parameters such as loxodrome distance, rhumb line and geographic and geocentric latitude of a position on the globe surface, in terms of the inverse incomplete elliptic integral of the third kind  $\Pi$ , discussing and surveying certain traditional and recent methods of computation of this function.

**Key Words:** Meridian (Loxodrome) sailing, Incomplete elliptic integral of the third kind, Numerical quadrature, Binomial series expansion of the integrand, Bulirsch method, Carlson's duplication, Fukushima's  $J$ -method.

## 1. INTRODUCTION

From the nautical point of view we have in the surface navigation two basic ship tracking models exist, when the trajectory is prescribed.

First, the constant course navigation (or loxodromic navigation), when in the unit time no course changes occur. Under this assumption we recognize two specific cases: **(i)** course line is a meridian (meridian sailing) and **(ii)** course line is parallel or coincides with the equator. Second, the navigation with variable rhumb when course line cuts meridians under variable angles pointwise, which is the orthodromic sailing.

We will focus here to loxodromic navigation and its specific case **(i)**. However, because the distance between two parallel lines is the same, regardless of the meridian along which it is measured, the distance  $L_m$  along any meridian between two parallels becomes a constant multiple  $L_{lox} = \sec \alpha \cdot L_m$  of the secant of the constant angle  $\alpha$  between the rhumb line and a meridian on the globe and the meridian distance  $L_m$ . Obviously, it is enough to study meridian sailing's length.

Loxodromic navigation is setting by a couple of relative coordinates

$$(\Delta\varphi, \Delta\lambda) = (\varphi_2 - \varphi_1, \lambda_2 - \lambda_1),$$

where  $\varphi_1, \varphi_2$ , that is  $\lambda_1, \lambda_2$  stand for the absolute geographic coordinates of latitude and longitude respectively and index 1 denotes the departure, and 2 the arrival positions. The meridian sailing corresponds by definition to  $\Delta\lambda = 0$ , while the sign of  $\Delta\varphi$  determines the general loxodromic course, which is  $K_L = 0^\circ$  for positive, and  $K_L = 180^\circ$  for negative  $\Delta\varphi$ .

In this article we study the meridian (loxodromic) arc length on a meridian on the globe spheroid surface starting at latitude  $\varphi_1$  and finishing route at latitude  $\varphi_2$ . Without losing generality we shall assume that  $\varphi_1 = 0$ , that is  $\Delta\varphi = \varphi_2 \equiv \varphi$  can be taken, i.e. the navigation

begins precisely on the equator. Indeed, we have  $\Delta\varphi = (\varphi_2 - \psi) - (\varphi_1 - \psi)$ ,  $\psi$  arbitrary, which reduces the meridian arc length calculation to the case  $\varphi_1 \equiv 0$ .

## 2. MERIDIAN SAILING ON ELLIPSOID OF REVOLUTION

Figure 1. presents the ellipsoid of revolution (two-axis or rotational ellipsoid) model of the globe with the equatorial radius  $a$  and polar radius  $b$ . The great ellipse is a planar cross-section of the ellipsoid and the plane containing the symmetry axis. (The associated numerical eccentricity we denote by the usual  $e^2 = 1 - b^2/a^2 \leq 1$ ). On the other hand  $\theta$  signifies the geocentric latitude of the same position  $P$ . Let us mention that  $\tan \theta = (1 - e^2) \tan \varphi$ .

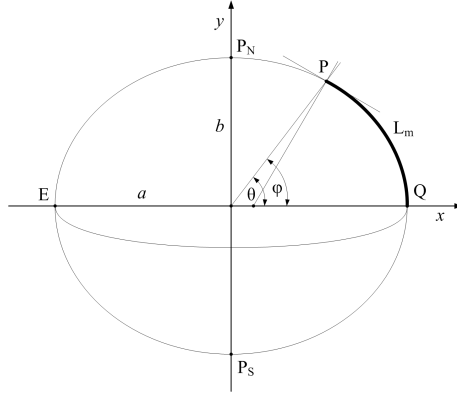


Figure 1. Meridian Ellipse on the Ellipsoid of Revolution.

The meridian ellipse arc length  $L_m$  characterizes the path between departure position  $Q$  at the equator and the arrival position  $P$  and reads as follows

$$L_m = a(1 - e^2) \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}} = \frac{b^2 \Pi(e^2; \varphi | e^2)}{a}, \quad (1)$$

where  $\Pi$  denotes the Legendre incomplete elliptic integral of the third kind defined by

$$\Pi(n; \varphi | m) := \int_0^\varphi \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - m \sin^2 t}} \quad |\varphi| \leq \frac{\pi}{2}, \quad (2)$$

see APPENDIX A. Here  $\varphi$  is the geographic latitude of  $P$ <sup>1</sup>, and for the sake of simplicity, here and in what follows, we write  $\Pi(\cdot) \equiv \Pi(e^2; \cdot | e^2)$ , see [41]. The special function  $\Pi$  is continuous and odd on the range  $|\varphi| \leq \pi/2$ , and being

$$\frac{d}{d\varphi} \Pi(n; \varphi | m) = \frac{1}{(1 - n \sin^2 \varphi) \sqrt{1 - m \sin^2 \varphi}} > 0 \quad n, m \in (0, 1),$$

<sup>1</sup>According to the remark upon rhumb line course  $\alpha \neq 90^\circ$  we have [45, p. 134, Eq. (6)] that  $L_{lox} = L_m \cdot \sec \alpha$ . Therefore

$$L_{lox} = \frac{L_m}{\cos \alpha} = \frac{b^2 \Pi(\varphi)}{a \cdot \cos \alpha}.$$

For  $\alpha = 90^\circ$  we have equator, or parallel sailing, while for  $\alpha = 0^\circ$  it is  $L_{lox} \equiv L_m$ .

it is monotone increasing function in  $\varphi$ . The monotonicity property implies invertibility, that is,  $w = \Pi(n; z|m)$  possesses an unique inverse  $z = \Pi^{-1}(n; w|m)$ , say. Now, by virtue of (1) we clearly conclude that the meridian distance between positions  $Q, P$  one corresponds to the latitude

$$\varphi = \Pi^{-1}\left(\frac{a L_m}{b^2}\right). \quad (3)$$

The citations regarding inverse elliptic integral of the third kind are very rare; Allender–Hornreich–Johnson [3, p. 2655] refer to Saupe [43, p. 817, Eq. (7)] concerning the exact solution for the approximated Euler–Lagrange equation which describes the free energy of the uniformly distorted Fréedericksz state, remarking that the exact solution of approximated EL–equation “... it is expressed ... as an inverse elliptic function of the third kind. This is inconvenient for numerical calculations...”. They considered binomial type approximations for  $\Pi^{-1}(w) \approx C \cdot (1 - w^\rho)$ ,  $\rho$  real,  $C$  an absolute constant (see [3, p. 2657]). However, Saupe followed the usual computation way exposed in the section 4.

Bearing in mind the fact that there is no explicit formula for  $\Pi$  and  $\Pi^{-1}$  expressed *via* certain elementary or higher transcendental functions, we are forced to compute it by numerical methods with prescribed accuracy. So, the meridian arc length  $L_m = L_m(\varphi)$  we compute by finding the associated  $\Pi$  function value in (1). Similarly, solving the inverse problem  $\Pi^{-1}$  in (3) we can calculate it only by certain numerical computation procedures, having computation error under the prescribed accuracy level. Let us remark that e.g. the computational softwares contain in–built subroutines such as `EllipticPi[n, z, m]` in *Mathematica*. In the next chapters we will survey these questions with a list of related links.

### 3. COMPUTING $\Pi(\varphi)$ BY NUMERICAL QUADRATURES

The computation of the meridian arc length  $L_m$ , one reduces to finding values of the elliptic integral of the third kind

$$\Pi(\varphi) = \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}}. \quad (4)$$

Based on polynomial interpolation of the integrand using uniform grid, we calculate  $\Pi(\varphi)$  by certain of the familiar numerical quadratures such as Newton–Cotes formulae, represented by composite rectangle and trapezoidal rules (constant and linear approximation), Simpson’s formula (quadratic polynomial approximation); Gaussian quadrature formula and its numerous variants e.g. Gauss–Legendre, Gauss–Kronrod, Gauss–Lobatto rules etc. For instance, the composite trapezoidal formula for  $\Pi(\varphi)$  is

$$\Pi(\varphi) = \frac{\varphi}{2n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 2 \sum_{j=1}^{n-1} \left[ \left( 1 - e^2 \sin^2 \left( \frac{\varphi j}{n} \right) \right)^{-3/2} \right] \right\} + R_n^T(\varphi), \quad (5)$$

where the error term

$$R_n^T(\varphi) = \frac{3e^2 \varphi^3 [4(e^2 - 2) \cos(2\vartheta\varphi) + e^2(2 \cos(4\vartheta\varphi) - 7)]}{32n^2(1 - e^2 \sin^2(\vartheta\varphi))^{7/2}}, \quad \vartheta \in (0, 1).$$

Knowing the computation accuracy  $\varepsilon > 0$ , the grids size  $n$  we can estimate from  $R_n(\varphi) < \varepsilon$ ,  $\varphi$  fixed. We mention that the composite Simpson's formula in our case reduces to

$$\begin{aligned} \Pi(\varphi) = \frac{\varphi}{6n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 4 \sum_{j=1}^{n-1} \left[ 1 - e^2 \sin^2 \left( \frac{\varphi j}{n} \right) \right]^{-3/2} \right. \\ \left. + 2 \sum_{j=1}^n \left[ 1 - e^2 \sin^2 \left( \frac{\varphi(2j-1)}{2n} \right) \right]^{-3/2} \right\} + R_n^S(\varphi), \end{aligned} \quad (6)$$

and the error term becomes

$$\begin{aligned} R_n^S(\varphi) &= \frac{\varphi^5}{2880 n^4} \left( (1 - e^2 \sin^2 t)^{-3/2} \right)' \Big|_{t=\xi} \\ &= \frac{3e^2 \varphi^5}{2880 n^4 (1 - e^2 \sin^2 \xi)^{11/2}} \left[ 11 e^6 \sin^8 \xi + 7.5 e^6 \sin^4(2\xi) - 18 e^4 \sin^6 \xi \right. \\ &\quad \left. + 2 \cos^2 \xi \cdot (52 e^6 \sin^6 \xi - e^4 \sin^4 \xi - 49 e^2 \sin^2 \xi - 2) \right. \\ &\quad \left. + 3e^2 \sin^4 \xi + 15 e^2 \cos^4 \xi \cdot (12 e^2 \sin^2 \xi + 1) + 4 \sin^2 \xi \right], \quad \xi \in (0, \varphi). \end{aligned}$$

Also by  $|R_n^S(\varphi)| < \varepsilon$  we evaluate  $n$  and than proceed to (6) with some fixed  $\varphi$ . Let us mention that these two numerical quadratures are in-built [29].

Transforming the integration interval  $[0, \varphi]$ ,  $\varphi \leq \pi/2$  onto  $[-1, 1]$  putting  $\varphi(t+1)/2 \mapsto t$  in the integrand of (4), the Gauss–Legendre quadrature rule gives the approximation

$$\begin{aligned} \Pi(\varphi) &= \frac{\varphi}{2} \int_{-1}^1 \left[ 1 - e^2 \sin^2 \left( \frac{\varphi}{2}(t+1) \right) \right]^{-3/2} dt \\ &= \frac{\varphi}{2} \sum_{j=1}^n w_j \left[ 1 - e^2 \sin^2 \left( \frac{\varphi}{2}(t_j+1) \right) \right]^{-3/2} + R_n^{GL}(\varphi), \end{aligned}$$

where the weight-coefficients

$$w_j = \frac{2}{(1 - t_j^2)[P_n'(t_j)]^2}, \quad j = \overline{1, n}$$

are based on the Legendre polynomial  $P_n(x)$ , see APPENDIX C. The related quadrature error formula turns out to be

$$R_n^{GL}(\varphi) = \frac{(n!)^4 \varphi^{2n+1}}{(2n+1)[(2n)!]^3} \left( (1 - e^2 \sin^2 t)^{-3/2} \right)^{(2n)} \Big|_{t=\xi}, \quad \xi \in (0, \varphi).$$

For another kind Gaussian quadratures, e.g Gauss–Kronrod quadrature, we refer to the original Kronrod's monograph [36]; also see the book [31] by Kahaner–Moler–Nash.

The Gauss–Lobatto quadrature is a kind of Gauss–Legendre quadrature procedure, since it is based on the Legendre polynomial  $P_n(x)$ , consult for more e.g. the celebrated work by Abramowitz–Stegun [1, pp. 888–900]. However, the related weights have been given up to  $n = 11$  in [15, pp. 63–64].

Gaussian quadrature is used in the QUADPACK library, the GNU Scientific Library, the NAG Numerical Libraries and statistical software *R*, see [28].

Also, we draw the mathematically better prepared reader's attention to the excellent monograph by Gil–Segura–Temme [24, §5.3: Gauss quadrature].

#### 4. BINOMIAL SERIES EXPANSION OF THE INTEGRAND

The most common computation procedure regards binomial series expansion of the integrand in (1) (see e.g. Saupe's paper [43]):

$$\begin{aligned} (1 - e^2 \sin^2 \varphi)^{-3/2} &= \sum_{n \geq 0} \binom{-3/2}{n} (-1)^n e^{2n} \sin^{2n} \varphi \\ &= 1 + \frac{3}{2} e^2 \sin^2 \varphi + \frac{15}{8} e^4 \sin^4 \varphi + \frac{35}{16} e^6 \sin^6 \varphi + \frac{315}{128} e^8 \sin^8 \varphi + \dots, \end{aligned} \quad (7)$$

the termwise integration applied after legitimate exchange the order of summation and integration gives the meridian distance  $L_m$  from the equator to a point  $P$  of latitude  $\varphi$  (the so-called Delambre expansion)

$$\begin{aligned} L_m = a(1 - e^2) \left\{ A\varphi - \frac{B}{2} \sin(2\varphi) + \frac{C}{4} \sin(4\varphi) \right. \\ \left. - \frac{D}{6} \sin(6\varphi) + \frac{E}{8} \sin(8\varphi) - \frac{F}{10} \sin(10\varphi) + \dots \right\}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} A &= 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11\,025}{16\,384} e^8 + \frac{43\,659}{65\,536} e^{10} + \dots \\ B &= \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2\,205}{2\,048} e^8 + \frac{72\,765}{65\,536} e^{10} + \dots \\ C &= \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2\,205}{4\,096} e^8 + \frac{10\,395}{16\,384} e^{10} + \dots \\ D &= \frac{35}{512} e^6 + \frac{315}{2\,048} e^8 + \frac{31\,185}{131\,072} e^{10} + \dots \\ E &= \frac{315}{16\,384} e^8 + \frac{3\,465}{65\,536} e^{10} + \dots \\ F &= \frac{693}{131\,072} e^{10} + \dots \end{aligned}$$

Higher order partial sums in the expansion (7) gives additional concretized terms in (8), see e.g. [34, p. 1, Eq. (1)]. Deakin [17] reported that for the Geodetic Reference System 1980 (GRS80) ellipsoid ( $a = 6\,378.137$  [km];  $b = 6\,356.752\,314\,140$  [km])<sup>2</sup> the coefficients

$$\begin{aligned} A &= 1.005\,052\,501\,813\,087, & B &= 0.005\,063\,108\,622\,224 \\ C &= 0.000\,010\,627\,590\,263, & D &= 0.000\,000\,020\,820\,379 \\ E &= 0.000\,000\,000\,039\,324, & F &= 0.000\,000\,000\,000\,071. \end{aligned}$$

<sup>2</sup>The currently used official WGS84 ellipsoid has equal  $a$ , but  $b = 6\,356.752\,314\,2$  [km]; this modest difference does not influenced the navigations calculations.

Therefore corresponding meridian arc length approximation formula, terms kept up to degree 10, reads

$$L_m^{80} = 6\,367.449\,139\,507\,109\,\varphi - 16.038\,508\,725\,491\,\sin(2\varphi) + 0.016\,832\,613\,310\,\sin(4\varphi) \\ - 0.000\,021\,984\,375\,\sin(6\varphi) + 3.1\,142 * 10^{-8}\,\sin(8\varphi) - 4.49 * 10^{-11}\,\sin(10\varphi). \quad (9)$$

Multiplying coefficients by  $1 - e^2$ , Lauf [37, p. 36] transformed formula (8) into

$$L_m = a \{A_0\varphi - A_2\sin(2\varphi) + A_4\sin(4\varphi) - A_6\sin(6\varphi) + A_8\sin(8\varphi) + \dots\}, \quad (10)$$

where

$$A_0 = 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16\,384}e^8 + \dots \approx 0.998\,324\,298\,423\,043 \\ A_2 = \frac{3}{8} \left( e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \dots \right) \approx 0.002\,514\,607\,124\,555 \\ A_4 = \frac{15}{256} \left( e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \dots \right) \approx 0.000\,002\,639\,111\,298 \\ A_6 = \frac{35}{3072} \left( e^6 + \frac{5}{4}e^8 + \dots \right) \approx 0.000\,000\,003\,446\,837 \\ A_8 = \frac{315}{131\,072} \left( e^8 + \dots \right) \approx 0.000\,000\,000\,004\,883 \quad \text{etc.}$$

Being ignored powers of  $e$  higher than  $e^6$  in (10) one obtains the official formula for meridian distance used by Geocentric Datum of Australia Technical Manual (ICSM 2002) [30]

$$\tilde{L}_m = 6\,367.449\,145\,771\,052\,\varphi - 16.038\,508\,741\,588\,\sin(2\varphi) \\ + 0.016\,832\,613\,417\,\sin(4\varphi) - 0.000\,021\,984\,399\,\sin(6\varphi), \quad (11)$$

which modestly differs from  $L_m^{80}$  expression; also see [37, p. 36, Eq. (3.55)]<sup>3</sup>.

In 1837 the celebrated scholar Bessel [7] presented a meridian arc length formula, expressed in terms of the third flattening (or oblateness) of the globe ellipsoid  $n = (a - b)/(a + b)$  (not equal to the Legendre modulus characteristic in  $\Pi(n; \varphi|m)$ ). He rewrote (8) into

$$L_m^B = a(1 - n)^2(1 + n) \left\{ \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 \right) \varphi - \left( \frac{3}{2}n + \frac{45}{16}n^3 + \frac{525}{128}n^5 \right) \sin(2\varphi) \right. \\ \left. + \left( \frac{15}{16}n^2 + \frac{105}{64}n^4 \right) \sin(4\varphi) - \left( \frac{35}{48}n^3 + \frac{945}{768}n^5 \right) \sin(6\varphi) \right\}, \quad (12)$$

The main advantage of (12) over the partial sum of order 10 of  $L_m$  in (8) is its good convergence, as  $n \approx e^2/4$  and Bessel's formula employed a reduced number of addends with approximately the same precision, see [34].

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<sup>3</sup>Deakin pointed out that at  $\varphi = 50^\circ$ , meridian distances calculated with (9) and by the formula  $\tilde{L}_m$  (11) differ for  $4.11 \cdot 10^{-4}$  [m], that is, "...GDA Technical Manual will give millimetre accuracy for latitudes covering Australia" [17].

The German geodesist Helmert [25, pp. 44–48] gave an approximation formula for  $L_m \approx L_m^H$  expressed *via* the same ellipsoid parameter  $n$ . Considering the series expansion <sup>4</sup>

$$\begin{aligned} L_m = a(1-n)(1-n^2) & \left\{ \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \dots \right) \varphi - \left( \frac{3}{2}n + \frac{45}{16}n^3 + \dots \right) \sin(2\varphi) \right. \\ & + \frac{1}{2} \left( \frac{15}{8}n^2 + \frac{105}{32}n^4 + \dots \right) \sin(4\varphi) - \frac{1}{3} \left( \frac{35}{16}n^3 + \dots \right) \sin(6\varphi) \\ & \left. + \frac{1}{4} \left( \frac{315}{128}n^4 + \dots \right) \sin(8\varphi) + \dots \right\}, \end{aligned}$$

Helmert multiplied coefficients by  $(1-n^2)^2$  and rejected all terms beginning with  $\sin(10\varphi)$  and making use of truncated coefficients excluding all terms involving powers of  $n$  greater than  $n^4$  achieving [18, p. 70, Eq. (226)], [34, p. 2, Eq. (6)]

$$\begin{aligned} L_m^H = \frac{a}{1+n} & \left\{ \left( 1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 \right) \varphi - \left( \frac{3}{2}n - \frac{3}{16}n^3 \right) \sin(2\varphi) \right. \\ & \left. + \left( \frac{15}{16}n^2 - \frac{15}{64}n^4 \right) \sin(4\varphi) - \frac{35}{48}n^3 \sin(6\varphi) + \frac{315}{512}n^4 \sin(8\varphi) \right\}. \end{aligned} \quad (13)$$

This alternative Helmert's formula is essentially the same as [5, p. 50, Eq. (5.5)].

A way to compute the latitude  $\varphi$  given a meridian arc length  $L_m$  solved Lauf [37, p. 38, Eq. (3.72)] by series inversion technique:

$$\begin{aligned} \varphi = \sigma + \left( \frac{3}{2}n - \frac{27}{32}n^3 + \dots \right) \sin(2\sigma) & + \left( \frac{21}{16}n^2 - \frac{55}{32}n^4 + \dots \right) \sin(4\sigma) \\ & + \left( \frac{151}{96}n^3 + \dots \right) \sin(6\sigma) + \left( \frac{1097}{512}n^4 - \dots \right) \sin(8\sigma) + \dots, \end{aligned}$$

where

$$\sigma = \frac{L_m}{G}, \quad G = a(1-n)(1-n^2) \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \dots \right).$$

Here  $G$  stands for the mean length of a meridian arc of one radian, see also [18, p. 71, Eq. (229)].

Applying numerical root-finding routines like Newton–Raphson, secant, regula falsi, iteration etc. methods to Helmert's formula (13), we can perform numerical latitude computation with given accuracy, consult Deakin–Hunter's Newton–Raphson iteration MATLAB routines [18, pp. 107–119, §3.2].

Belyakov–Kravtsova–Rappaport [6, p. 5] used two standard power series in  $n$  and  $m$  to evaluate  $\Pi(n; \varphi | m)$ . Following their traces, and bearing in mind that we take  $n = m = e^2$ , their power series coincide with

$$\Pi(\varphi) = \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} A(j) \left( \frac{e}{2} \right)^{2j} \quad (14)$$

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<sup>4</sup>This series expansion in terms of  $n$  possesses a faster rate of convergence than the meridian distance formula (8) involving powers of  $e^2$ .

$$= \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} B(j) \left( -\frac{b^2}{4a^2} \right)^j. \quad (15)$$

where

$$A(j) = \int_0^\varphi \frac{\sin^{2j} t}{1 - e^2 \sin^2 t} dt, \quad B(j) = \int_0^\varphi \frac{\sec t \tan^{2j} t}{1 - e^2 \sin^2 t} dt.$$

Because series (14) is not useful for numerical computation for  $e|\sin \varphi|$  near to 1 and series (15) slowly converges for  $ba^{-1}|\tan \varphi|$  close to 1<sup>5</sup>, Franke [20] reformulated the method by Ai–Harrison [2] to evaluate  $\Pi(\varphi)$  for a range of  $\varphi$  critical for the direct use of series (14), (15); he introduced the so-called "double-angle" approach remarking that "... *one application is sufficient for computational purposes*", see [20, p. 495].

We point out that by some longer routine calculation one arrives at

$$A(j) = \frac{\sin^{2j+1} \varphi}{2j+1} F_1 \left( j + \frac{1}{2}; j+1, 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi \right) \quad (16)$$

$$B(j) = \frac{\sin^{2j+1} \varphi}{2j+1} F_1 \left( j + \frac{1}{2}; \frac{1}{2}, 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi \right), \quad (17)$$

where both Appell's hypergeometric  $F_1$ -functions of two variables converge, being  $\Pi(\varphi)$  incomplete elliptic integral, that is,  $|\varphi| < \pi/2$ , consult APPENDIX B. Let us mention that  $F_1$  is in-built in *Mathematica*, having code `AppellF1[a, b_1, b_2, c, x, y]`, which evaluate  $F_1$ -values with the desired numerical precision inside the domain  $|x| < 1, |y| < 1$ . Now, approximating  $\Pi(\varphi)$  by suitably long partial sum of series (14), (15) in conjunction with (16), (17), we conclude a new appropriate numerical evaluation tool.

Finally, the convergence rate problem obviously occurs caused by slowly convergent binomial series (7) and the subsequent trigonometric series concluded by termwise integration. So, these methods seem to be inefficient for general input argument values [12, 21]. Moreover, preliminary trials show that the approach using the theta functions [1], is also inferior to the computation procedures applying Gauss or Bartky transformations [21, p. 1963].

## 5. POWER SERIES EXPANSIONS

The Maclaurin series expansion of the integrand (7) leads to

$$\Pi(\varphi) = \varphi + \frac{3e^2}{2}\varphi^3 + \left( \frac{3e^4}{8} - \frac{e^2}{10} \right) \varphi^5 + \left( \frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105} \right) \varphi^7 + R_8^M(\varphi), \quad (18)$$

where the error term in the Lagrange form reads

$$R_8^M(\varphi) = \frac{\Pi^{(8)}(\vartheta\varphi)}{40320} \varphi^8, \quad \vartheta \in (0, 1).$$

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<sup>5</sup>Actually, computational problems occur already for  $ba^{-1}|\tan \varphi| \geq 0.7$ , [20, p. 495].



The related Maclaurin polynomial gives the low order, therefore expectedly not to precise, approximation formula

$$L_m^M = \frac{b^2}{a} \left\{ \varphi + \frac{3e^2}{2} \varphi^3 + \left( \frac{3e^4}{8} - \frac{e^2}{10} \right) \varphi^5 + \left( \frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105} \right) \varphi^7 \right\}, \quad (19)$$

which turns out to be the truncated partial sum of the power series expansion presented in APPENDIX B - up to the constant  $b^2/a$ . Obviously, close-to-nil  $\varphi$  guarantees higher precision employing (19).

Completely different kind series expansion results are established by Karp–Savenkova–Sitnik [32, 33]. They considered  $\Pi(\varphi)$  on the range

$$\varphi \in \mathcal{D} = \left[ -\max \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\}, \min \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\} \right],$$

see [32, p. 334, Theorem 1]. Related worked examples can be found in [33].

## 6. BULIRSCH, AND FUKUSHIMA–ISHIZAKI METHODS

Bulirsch introduced a new way to compute all three elliptic integrals  $F, E, \Pi$  in both complete and incomplete cases. (However, we concentrate here only to his results regarding elliptic integral of the third kind). His approach is based on the Bartky's transformation (actually a by-product of the familiar Landen, or Gauss–Landen transformation) [10]. Bulirsch gave the routine:

$$el3(x, k_c, p) = \int_0^{\arctan x} \frac{d\gamma}{(\cos^2 \gamma + p \sin^2 \gamma) \sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1, \quad (20)$$

where the main branch of the arctan is used, that is,  $|\arctan x| \leq \pi/2$ , and according to the notations introduced in (4),  $x, k_c, p$  correspond to  $\tan \varphi, \sqrt{1 - e^2}, 1 - e^2$  respectively. Cosequently, in our setting written (compare APPENDIX A):

$$\Pi(\varphi) = el3(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2).$$

The ALGOL procedure for  $el3$  is available from [11].

Having cancellation problems with the complementary characteristic  $p = 1 - n$  in  $el3$ , when parameter  $|n|$  of  $\Pi(n; \varphi|m)$  is small, Bulirsch unified  $el3$  by [11, p. 279, Eq. (4.1.14)]

$$el(x, k_c, p, a, b) = \int_0^{\arctan x} \frac{(a \cos^2 t + b \sin^2 t) d\gamma}{(\cos^2 \gamma + p \sin^2 \gamma) \sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1.$$

Moreover, he didn't gave related computation routine. Fukushima–Ishizaki extended  $el(\cdot)$  to the 'incomplete' routine function [23, p. 240, Eq. (21)]

$$G(\varphi, n_c, m_c, a, b) = \int_0^\varphi \frac{(a \cos^2 \gamma + b \sin^2 \gamma) d\gamma}{(\cos^2 \gamma + n_c \sin^2 \gamma) \sqrt{\cos^2 \gamma + m_c \sin^2 \gamma}}, \quad (21)$$

where parameters  $n_c, m_c$  have been introduced to avoid a loss of precision. Hence, they solve the cancellation problem with respect to  $n$  positively [23].

Obviouly (consult APPENDIX A)

$$\Pi(\varphi) = G(\varphi, 1 - e^2, 1 - e^2, 1, 1).$$

Complete description of the Fukushima–Ishizaki routine and the calculation algorithm is presented with companion numerical examples in [23].

## 7. DUPLICATION METHOD BY CARLSON

We have mention that Franke initiates a kind of duplication of angles in computing  $\Pi(\varphi)$  with a desired accuracy, [20]. In a set of articles, see [13] and his references [3, 4, 5, 6] therein, Carlson developed a new approximation method by introducing a new set of so-called  $R$ -integrals in the standardized form. Concentrating to  $\Pi(\varphi)$ , the adequate standard Carlson integrals of the first and the third kind are [13, p. 2, Eqs. (1.1),(1.3)]

$$\begin{aligned} R_F(x, y, z) &= \frac{1}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} dt \\ R_J(x, y, z, \rho) &= \frac{3}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} (t+\rho)^{-1} dt, \quad \rho \neq 0. \end{aligned}$$

These functions are symmetric in  $x, y, z$  and normalized so, that  $R_F(x, x, x) = x^{-1/2}$ , while  $R_J(x, x, x, x) = x^{-3/2}$  and  $R_J(0, y, z, \rho)$  is said to be complete. Two cases are signified as

$$R_C(x, y) = R_F(x, y, y), \quad R_D(x, y, z) = R_J(x, y, z, z). \quad (22)$$

The variables  $x, y, z$  are nonnegative and at most one of them is zero<sup>6</sup>. The link to  $\Pi(\varphi)$  is [13, p. 8, Eq. (4.3)]

$$\Pi(\varphi) = \sin \varphi \cdot R_F(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1) + \frac{e^2}{3} \sin^3 \varphi \cdot R_D(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1). \quad (23)$$

The connection with Bulirsch *el3* procedure becomes

$$\begin{aligned} \Pi(\varphi) &= el3(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2) = \tan \varphi \cdot R_F\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right) \\ &\quad + \frac{1 - e^2}{3} \tan^3 \varphi \cdot R_D\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right). \end{aligned}$$

Bearing in mind (23), we have to expose Carlson's computation procedure for the duplication formulae [14, Eqs. 19.26.18–19, 22–23]

$$R_F(x, y, z) = 2R_F(x + \lambda, y + \lambda, z + \lambda) = R_F\left(\frac{x + \lambda}{4}, \frac{y + \lambda}{4}, \frac{z + \lambda}{4}\right) \quad (24)$$

$$R_D(x, y, z) = 2R_D(x + \lambda, y + \lambda, z + \lambda) + \frac{3}{\sqrt{z(z + \lambda)}}, \quad (25)$$

where

$$\lambda = \sqrt{xy} + \sqrt{xz} + \sqrt{yz}.$$

<sup>6</sup>We have to point out that Carlson's set of  $R$ -integrals cover a wide family of logarithmic, inverse circular, inverse hyperbolic functions, complete and incomplete elliptic integrals as special cases [13, §4].

After the difference between the variables have been made enough small by successive duplications by (24) and (25), the integrals  $R_F, R_D$  are expanded into a triple Maclaurin series and being approximated by their partial sums of order five. The resulting algorithms consist only from rational operations and square roots. The convergence is fast enough to be superior with methods possessing second order convergence, unless extremely high precision is required, [13, p. 2]. The related algorithms for  $R_F, R_C, R_J, R_D$  are given respectively in [13, pp. 2–5, **Algorithm 1-4.**], but they are too massive to be presented here in detail.

## 8. $J$ -FUNCTION METHOD BY FUKUSHIMA.

Leaded by certain practical reasons such as avoiding the loss of significant digits for small  $|n|$ , Fukushima [21] introduced a linear combination of the Legendre's incomplete elliptic integrals of the first (see APPENDIX A) and third kind instead of  $\Pi(n; \varphi|m)$  itself:

$$J(n; \varphi|m) = \frac{1}{n} (\Pi(n; \varphi|m) - F(\varphi|m)) \quad \varphi \in \left(0, \frac{\pi}{2}\right), m, n \in (0, 1),$$

calling it associate incomplete elliptic integral of the third kind. Obviously, knowing  $J(n; \varphi|m)$  and  $F(\varphi|m)$  we have

$$\Pi(n; \varphi|m) = F(\varphi|m) + nJ(n; \varphi|m).$$

Fukushima outlined [21, p. 1963, §1.3] that the standard computation methods employed to  $J(n; \varphi|m)$  are Bulirsch's, and Carlson's (applied to  $R_J$ ). However, from the practical point of view both methods suffer due to either cancellation errors or large computational amount, see the discussions around [21, p. 1964, Eqs. (19), (21)]; the comparison to Bulirsch's and Carlson's methods are given in [21, §4].

Fukushima's  $J$ -method is based on selecting one function from the four possible expressions:

$$J(n; \varphi|m) = \begin{cases} J(n; \varphi|m) & \text{(i)} \\ J(n|m) - J\left(n; \arcsin \frac{\cos \varphi}{\sqrt{1 - n \sin^2 \varphi}} \middle| m\right) - R_C(t_s^{-2}, h + t_s^{-2}) & \text{(ii)} \\ J(n|m) - J\left(n; \arccos \sqrt{\frac{1 - m}{1 - m \sin^2 \varphi}} \middle| m\right) - R_C(t_c^{-2}, h + t_c^{-2}) & \text{else} \end{cases}$$

where  $J(n|m) = J(n; \pi/2|m)$  is the associate complete elliptic integral of the third kind and

$$\begin{cases} \text{(i)} & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{\sqrt{1+m}+1}}, \\ \text{(ii)} & \frac{\cos \varphi}{\sqrt{1 - n \sin^2 \varphi}} < y_s \end{cases},$$

while

$$h = n(1 - n)(n - m), \quad t_s = \frac{\sin(2\varphi)}{2\sqrt{1 - m \sin^2 \varphi}}, \quad t_c = \frac{t_s}{1 - n}.$$

Also,  $y_s, \varphi_s$  are constants, where  $y_s \in \{0.95, 0.90\}$  in the single and double precision computation modalities respectively, while  $\varphi_s = \arcsin \sqrt{y_s} \in \{1.345, 1.249\}$ .

In our environment writing  $J(\varphi) \equiv J(e^2; \varphi|e^2)$ ,  $J(e^2) \equiv J(\pi/2)$ ;  $F(\varphi) = F(\varphi|e^2)$ , we get the associate incomplete elliptic integral of the third kind

$$J(\varphi) = \frac{1}{n} (\Pi(\varphi) - F(\varphi)), \quad (26)$$

and the associated mathematical model

$$J(\varphi) = \begin{cases} J(\varphi) & \text{(i)} \\ J(e^2) - J\left(\arcsin \frac{\cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2\sqrt{1 - e^2 \sin^2 \varphi}} & \text{(ii)} \\ J(e^2) - J\left(\arccos \frac{b}{a\sqrt{1 - e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2(1 - e^2)\sqrt{1 - e^2 \sin^2 \varphi}} & \text{otherwise} \end{cases} \quad (27)$$

for

$$\begin{cases} \text{(i)} & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{\sqrt{1 + e^2} + 1}} \\ \text{(ii)} & \frac{\cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} < y_s \end{cases}.$$

Indeed,  $h = 0$  and  $R_C(x, x) = R_F(x, x, x) = x^{-1/2}$ , according to (22). Now, it remains to apply Fukushima's half and double argument transformations [21, pp. 1966–1970, §3.3–3.7] to the model (26) to earn the desired value of  $J(\varphi)$  for  $\varphi$  already given.

## 9. LUKE'S PADÉ APPROXIMATION METHOD

Using Padé <sup>7</sup> approximations for the square root, Luke [38, 39] offered routines for approximative evaluation of elliptic integrals of all three kinds. His expression looks like (in our setting  $n = m = e^2$ ) [39, p. 193, Eqs. (26)]:

$$\begin{aligned} \Pi(\varphi) &= \frac{H(\varphi, e^2)}{2n + 1} \left\{ 1 + \sum_{j=1}^n \frac{2}{\cos^2 \frac{j\pi}{2n+1}} \right\} \\ &\quad - \frac{2}{2n + 1} \sum_{j=1}^n H\left(\varphi, e^2 \sin^2 \frac{j\pi}{2n + 1}\right) \tan^2 \frac{\pi j}{2n + 1} + R_n^L(\varphi) \\ H(\tau, f) &= \frac{a}{b} \arctan \left( \frac{b}{a} \tan \tau \right) \\ R_n^L(\varphi) &= \frac{2e^{-(2n+1)\zeta} \tan \varphi}{(1 - e^2 \sin^2 \varphi)(4n + 3)} \left\{ 1 + \frac{\lambda_1}{4n + 3} + \mathcal{O}(n^{-2}) \right\} + \mathcal{O}(e^{-4n\zeta}) \\ \lambda_1 &= 1 - \frac{\cosh \zeta - 1}{\sinh \zeta \cdot \cos^2 \varphi} - \frac{2e^2 \sin^2 \varphi \cdot (\cosh \zeta - 1)}{(1 - e^2 \sin^2 \varphi) \sinh \zeta}. \end{aligned}$$

<sup>7</sup>Henry Eugène Padé (1863–1953) a French mathematician who made important contributions to the theory of continued fractions and introduced what we call today Padé approximants, which are rational approximations to functions given by their power series.

Here  $R_n^L(\varphi)$  is the error term which vanishes with growing  $n$  [39, p. 194, Eq. (28)]:

$$\lim_{n \rightarrow \infty} R_n^L(\varphi) = 0,$$

and  $e^\zeta$  is defined by [39, p. 191]

$$e^\zeta = \frac{2 - e^2 \sin^2 \varphi + 2\sqrt{1 - e^2 \sin^2 \varphi}}{e^2 \sin^2 \varphi}.$$

We are drawing the reader's attention to the important difference in notations of the base of natural logarithms  $e$ , and the eccentricity  $e$ .

A further variant of this method, useful for  $\varphi$  near to  $\pi/2$ , has proposed in [39, pp. 194–196, Eqs. (29)–(40)].

## 10. COMPARISON OF EXPOSED METHODS

Illustration of the presented methods will be the comparison of meridian distance calculation's efficiency of composite trapezoidal (5) and Simpson's (6) rules for few subdivisions  $n$ , Geodetic Reference System 1980 (GRS80) ellipsoid formula  $L_m^{80}$  (9), Geocentric Datum of Australia Technical Manual (ICSM 2002) official formula  $\tilde{L}_m$  (11), Bessel's  $L_m^B$  (12), Helmert's  $L_m^H$  (13) formulae, and the power series approximation formula  $L_m^M$  (18), using referent values of WGS84 ellipsoid characteristics  $a = 6378.137 [km]$ ,  $e = 0.0818192$  and the latitude of Senj <sup>8</sup>  $\varphi(S) = 45^\circ$  N. The corresponding value

$$\Pi(\pi/4) = \text{EllipticPi}[0.006694381, \pi/4, 0.006694381] = 0.786834838411$$

was evaluated by the **WolframAlpha** computational engine. By (1) we get <sup>9</sup>

$$L_m = a(1 - e^2)\Pi(\pi/4) = 4984.944374286 [km] = 2691.654630 [M].$$

First, by obvious rough estimates  $1 - e^2 \sin^2(\vartheta\varphi) \geq \cos^2(\vartheta\varphi) \geq \cos^2 \varphi$ , which hold true for all  $\varphi \in (0, \pi/2)$ ,  $\vartheta \in (0, 1)$ , we conclude that

$$\begin{aligned} |R_n^T(\pi/4)| &\leq \left. \frac{3e^2(13e^2 + 8)\varphi^3}{32n^2 \cos^7 \varphi} \right|_{\varphi=\frac{\pi}{4}} = \frac{0.003430}{n^2} \\ |R_n^S(\pi/4)| &\leq \left. \frac{e^2\varphi^5(122.5e^6 + 200e^4 + 116e^2 + 4)}{960n^4 \cos^{11} \varphi} \right|_{\varphi=\frac{\pi}{4}} = \frac{0.000828557}{n^4}. \end{aligned}$$

Denote  $L_m^T, L_m^S$  the right-hand-side sums in the trapezoidal (5) and Simpson's (6) rules, respectively. We get the following table

<sup>8</sup>Senj is a small town situated on the Adriatic coast nearby Rijeka, Croatia.

<sup>9</sup>Here and in what follows, all computed values of meridian distances are expressed in  $[km]$  units; by  $L_m [km] = L_m/1.852 [M]$  we clearly transform it into nautical miles.

| $n$ | $L_m^T$      | $ R_n^T  <$     | $L_m^S$      | $ R_m^s  <$      |
|-----|--------------|-----------------|--------------|------------------|
| 2   | 4985.777 412 | $8.6 * 10^{-4}$ | 4984.907 385 | $5.2 * 10^{-5}$  |
| 3   | 4985.312 473 | $3.8 * 10^{-4}$ | -            | -                |
| 4   | 4985.151 015 | $2.1 * 10^{-4}$ | 4984.942 218 | $3.2 * 10^{-6}$  |
| 5   | 4985.076 504 | $1.3 * 10^{-4}$ | -            | -                |
| 6   | 4985.036 084 | $9.5 * 10^{-5}$ | 4984.943 954 | $6.4 * 10^{-7}$  |
| 10  | 4984.977 367 | $3.4 * 10^{-5}$ | 4984.944 321 | $8.3 * 10^{-8}$  |
| 20  | 4984.952 620 | $8.6 * 10^{-6}$ | 4984.944 372 | $5.2 * 10^{-9}$  |
| 50  | 4984.945 696 | $1.4 * 10^{-6}$ | 4984.944 372 | $1.3 * 10^{-10}$ |

Table 1. TRAPEZOIDAL AND SIMPSON'S RULES CALCULATIONS OF MERIDIAN DISTANCES

Next, we approximate  $L_m$  by  $L_m^{80}$ ,  $\tilde{L}_m$ ,  $L_m^B$  and  $L_m^H$  by quoted formulae (9), (11), (12) and (13) respectively. First, the third flattening parameter of the globe WGS84 ellipsoid is equal to

$$n = \frac{a-b}{a+b} = 0.001\,679\,220.$$

This value, together with the associated eccentricity  $e$  transforms Bessel's, Helmert's and the Maclaurin expansion formulae into

$$\begin{aligned} L_m^B &= 6\,367.449\,148\,277\,502\,\varphi - 16.038\,504\,978\,751\,\sin(2\varphi) \\ &\quad + 0.016\,832\,605\,426\,\sin(4\varphi) - 0.000\,021\,984\,392\,\sin(6\varphi) \\ L_m^H &= 6\,367.449\,148\,277\,505\,\varphi - 16.038\,504\,978\,753\,\sin(2\varphi) \\ &\quad + 0.016\,832\,605\,523\,\sin(4\varphi) - 0.000\,021\,984\,408\,\sin(6\varphi) \\ &\quad + 0.000\,000\,031\,148\,\sin(8\varphi) \\ L_m^M &= 6\,335.432\,364\,595\,861\,\varphi + 63.617\,697\,072\,503\,\varphi^3 \\ &\quad - 2.999\,026\,589\,075\,\varphi^5 + 0.353\,815\,715\,411\,\varphi^7. \end{aligned}$$

Latitude  $\varphi = 45^\circ$  of Senj gives

$$\begin{aligned} L_m^{80} &= 4\,984.944\,373 \quad [km], & \tilde{L}_m &= 4\,984.944\,378 \quad [km] \\ L_m^B = L_m^H &= 4\,984.944\,384 \quad [km], & L_m^M &= 5\,005.826\,977 \quad [km]. \end{aligned}$$

Now, it is not hard to see all estimated values of  $L_m$  differ in 12 [mm]; exceptions are the composite trapezoidal formula with approximately 1.3 [m] deviation, and the very slowly convergent Maclaurin polynomial approximant  $L_m^M$  with near to 21 [km] overlength. Therefore we conclude that all exposed and successfully examined formulae have the same navigational and geodesical significance and applicability.

## 11. CONCLUSION

In this article we exposed without further using computation methods by Burlisch, Carlson, Fukushima–Ishizaki and Fukushima, since they are superior to the classical series expansion computation methods for the incomplete elliptic integral of the third kind  $\Pi(\varphi)$ .

The use of all presented methods is highly appreciated in numerous applications, because  $\Pi(n; \varphi|m)$  appears in various mathematical models such as the approximated Euler–Lagrange equation [3, p. 2655], [43, p. 817, Eq. (7)], the model of the graviational or electromagnetic field associated with scalar or vector potential of a simple distribution such as annular disks with finite thickness [21]; magnetic field caused by thick coil [19]; torque–free rotation of triaxial rigid body [22, Appendix C] and the periodic solutions of the Schrödinger equation [16].

Our main purpose here was to present enough precise computational tools and procedures for navigational calculations in both nautical and geodesical topics. Regarding nautical themata, has to be mentioned the very recent article by Weintrit–Kopacz [44] in which authors gave exhaustive up-to-date presentation, together with the numerous articles therein e.g. [46, 47]. We point out that all these articles work with formulae and routines in terms of the geographic latitude  $\varphi$ , the exception is Petrović’s note [42], where the meridian arc length has been expressed *via* the incomplete elliptic integral of the second kind as  $L_m = a \cdot E(\theta|e^2)$ , see [42, p. 88, Eq. (21)], also see the paper by Kos–Filjar–Hess [35, p. 959].

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## References

- [1] Abramowitz, M., Stegun, I.A. (Eds.) (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. 9th printing. Dover, New York.
- [2] Ai, D.K., Harrison, Z.L. (1964). A computer method for calculating the complete and incomplete elliptic integrals of the third kind. Report AD-437-207. California Institute of Technology.
- [3] Allender, D.W. Hornreich, R.M., Johnson D.L. (1987). Theory of the stripe in bend-Fréedericksz-geometry nematic films. *Physical Review Letters* **59**, No. 23, 2654–2657.
- [4] Andrews, G.E., Askey, R., Roy, R. (1999). *Special Functions*. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge.
- [5] Baeschlin, C.F. (1948). *Lehrbuch Der Geodäsie*. Orell Füssli Verlag, Zürich.
- [6] Belyakov, V.M., Kravtsova, P.I., Rappoport, M.G. (1965). *Tables of Elliptical Integrals*. Part I. Mathematical Tables Series, Vol. 37. A Pergamon Press Book The Macmillan Co., New York.
- [7] Bessel, F.W. (1837). Bestimmung der Axen des elliptischen Rotations sphäroids, welches den vorhandenen Messungen der Meridianbögen der Erde am meistens entspricht. *Astronomische Nachrichten* **14**, 333–346.
- [8] Bulirsch, R. (1965). Numerical calculation of elliptic integrals and elliptic functions. *Numerische Mathematik* **7**, 78–90.

- [9] Bulirsch, R. (1965). Numerical calculation of elliptic integrals and elliptic functions II. *Numerische Mathematik* **7**, 353–354.
- [10] Bulirsch, R. (1969). An extension of Bartky–transformation to incomplete elliptic integrals of the third kind. *Numerische Mathematik* **13**, 266–284.
- [11] Bulirsch, R. (1969). Numerical calculation of elliptic integrals and elliptic functions III. *Numerische Mathematik* **13**, 305–315.
- [12] Byrd, P.F., Friedman, M.D. (1971). *Handbook on Elliptic Integrals for Engineers and Physicists*. 2nd ed. Springer–Verlag, Berlin.
- [13] Carlson, B.C. (1979). Computing elliptic integrals by duplication. *Numerische Mathematik* **33**, 1–16.
- [14] Carlson, B.C. (2010). Chapter 19. *Elliptic Integrals*. In Olver, F.W.J., Lozier, D.W., Boisvert, R.F., Clark, C.W. (Eds.) *Handbook of Mathematical Functions*. Cambridge University Press, Cambridge. Available at [<http://dlmf.nist.gov/19>]
- [15] Chandrasekhar, S. (1960). *Radiative Transfer*. Dover, New York.
- [16] Chowa, K.W., Ng, T.W. (2011). Periodic solutions of a derivative nonlinear Schrödinger equation, elliptic integrals of the third kind. *Journal of Computational and Applied Mathematics* **235**, 3825–3830.
- [17] Deakin, R.E. (2006). Meridian distance. Unpublished manuscript, 30pp. Available online at [<http://user.gs.umat.edu.au/rod/files/publications>]
- [18] Deakin, R.E., Hunter, M.N. (2010). *Geometric Geodesy. Part A*. Second printing. RMIT University Melbourne, Australia.
- [19] Delinchant, B., Wurtz, F., Yonnet, J.-P., Coulomb, J.-L. (2011). Interaction between ring–shaped permanent magnets with symbolic gradients; application to magnetic bearing system optimization. *IEEE Transactions on Magnetics* **47**, 1418–1421.
- [20] Franke, C.H. (1965). Numerical evaluation of the elliptic integral of the third kind. *Mathematics of Computation* **19**, 494–496.
- [21] Fukushima, T. (2012). Precise and fast computation of a general incomplete elliptic integral of third kind by half and double argument transformations. *Journal of Computational and Applied Mathematics* **236**, 1961–1975.
- [22] Fukushima, T. (2008). Simple, regular and efficient numerical integration of rotational motion. *Astronomical Journal* **135**, 2298–2322.
- [23] Fukushima, T., Ishizaki, H. (1994). Numerical computation of incomplete elliptic integrals of a general form. *Celestial Mechanics and Dynamical Astronomy* **59**, 237–251.
- [24] Gil, A., Segura, J., Temme, N.M. (2007). *Numerical Methods for Special Functions*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA.
- [25] Helmert, F.R. (1880). *Die mathematischen und physikalischen Theorien der höheren Geodäsie. I. Teil. Die mathematischen Theorien*. Druck und Verlag B.G. Teubner, Leipzig.
- [26] <http://functions.wolfram.com/EllipticIntegrals/EllipticPi3/03/01/01/0006>
- [27] <http://functions.wolfram.com/EllipticIntegrals/EllipticPi3/06/01/07/0002>
- [28] <http://stat.ethz.ch/R-manual/R-patched/library/stats/html/integrate.html>
- [29] <http://www.zweigmedia.com/RealWorld/integral/integral.html>
- [30] ICSM. (2002). Geocentric Datum of Australia Technical Manual. Intergovernmental Committee on Surveying & Mapping (ICSM), Version 2.2, February 2002. [[www.anzlic.org.au/icsm/gdatum](http://www.anzlic.org.au/icsm/gdatum)]
- [31] Kahaner, D., Moler, C., Nash, S. (1989). *Numerical Methods and Software*. Prentice-Hall, Englewood Cliffs, NJ.
- [32] Karp, D., Savenkova, A., Sitnik, S.M. (2007). Series expansions for the third incomplete elliptic integral via partial fraction decompositions. *Journal of Computational and Applied Mathematics* **207**, No. 2, 331–337.
- [33] Karp, D., Savenkova, A., Sitnik, S.M. (2006). Series expansions for the third incomplete elliptic integral via partial fraction decompositions. [<http://arxiv.org/abs/math/0410009>], 11pp.
- [34] Kawase, K. (2011). A general formula for calculating meridian arc length and its application to coordinate conversion in the Gauss–Krüger projection. *Bulletin of the Geospatial Information Authority of Japan* **59**, December, 1–13.



- [35] Kos, S., Filjar, R., Hess, M. (2009). Differential equation of the loxodrome on a rotational surface. *Proceedings of the 2009 International Technical Meeting of the Institute of Navigation*. Anaheim, CA January 2009, 958–960.
- [36] Kronrod, A.S. (1965). *Nodes and Weights of Quadrature Formulas. Sixteen-place Tables*. Authorized translation from the Russian. Consultants Bureau, New York.
- [37] Lauf, G.B. (1983). *Geodesy and Map Projections*. 2nd ed. TAFE Publications Unit, Collingwood, Victoria, Australia.
- [38] Luke, Y.L. (1968). Approximations for elliptic integrals. *Mathematics of Computation* **22** 627–634.
- [39] Luke, Y.L. (1970). Further approximations for elliptic integrals. *Mathematics of Computation* **24** 191–198.
- [40] Mathai, A.M., Saxena, R.K. (1978). *The H-function with Applications in Statistics and other Disciplines*. Wiley Eastern Limited, New Delhi · Bangalore · Bombay.
- [41] Meyer, T.H., Rollins, C. (2011). The direct and indirect problem for loxodromes. *Navigation* **58**, No. 1, 1–6.
- [42] Petrović, M. (2007). Differential equation of a loxodrome on the spheroid. *Naše More* **54**, No. 3–4, 87–89.
- [43] Saupe, A. (1960). Die Biegeelastizität der nematischen Phase von Azoxyanisol. *Zeitschrift für Naturforschung* **15**, 815–822.
- [44] Weintrit, A., Kopacz, P. (2011). Geodesic based trajectories in navigation. *Coordinates* **VII**, No. 6. [<http://mycoordinates.org/geodesic-based-trajectories-in-navigation>]
- [45] Williams, J.E.D. (1950). Loxodromic distances on terrestrial spheroid. *The Journal of Navigation* **3**, No. 2, 133–140.
- [46] Williams, R., Phythian, J.E. (1989). Navigating along geodesic paths on the surface of a spheroid. *The Journal of Navigation* **42**, No. 1, 129–136.
- [47] Williams, R., Phythian, J.E. (1992). The shortest distance between two nearly antipodean points on the surface of a spheroid. *The Journal of Navigation* **45**, No. 1, 114–125.

APPENDIX A. To express  $L_m$  explicitly, we employ the Legendre’s incomplete elliptic integrals of the first, second and third kind defined respectively by

$$\begin{aligned}
 F(z|m) &= \int_0^z \frac{dt}{\sqrt{1-m\sin^2 t}} \\
 E(z|m) &= \int_0^z \sqrt{1-m\sin^2 t} dt \\
 \Pi(n; z|m) &= \int_0^z \frac{dt}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}} \quad |z| \leq \frac{\pi}{2};
 \end{aligned}$$

related functions

$$F\left(\frac{\pi}{2}\middle|m\right) = F(m), \quad E\left(\frac{\pi}{2}\middle|m\right) = E(m), \quad \Pi\left(n; \frac{\pi}{2}\middle|m\right) = \Pi(n|m)$$

signify the complete elliptic integrals of all three kinds. Also, function  $\Pi$  is odd and quasi-periodic, so there hold true properties [26]

$$\begin{aligned}
 \Pi(n; z|m) &= -\Pi(n; -z|m) \\
 \Pi(n; z+k\pi|m) &= \Pi(n; z|m) + k\Pi(n|m), \quad k \in \mathbb{Z}, |n| \leq 1 \\
 \Pi(0; z|m) &= F(z|m)
 \end{aligned}$$

$$\Pi(n; z|n) = \frac{1}{1-n} \left\{ E(z|n) - \frac{n \sin(2z)}{2\sqrt{1-n \sin^2 z}} \right\}.$$

The transcendental equation  $\Pi(n; z|n) = w$  has no explicit solution in  $z$ .

Rewriting (2) into

$$\Pi(n; \varphi|m) := \int_0^\varphi \frac{(\sin^2 t + \cos^2 t) dt}{(\cos^2 t + (1-n) \sin^2 t) \sqrt{\cos^2 t + (1-m) \sin^2 t}},$$

then introducing general parameters  $p = 1 - n, k_c^2 = 1 - m, x = \tan \varphi$ , Burilich in [8, 9, 10, 11], subsequently  $a, b, n_c = 1 - n, m_c = 1 - m$ , Fukushima–Ishizaki in [23] have proposed more general and redirected computation routines  $el3$  (20) and  $G$  (21) respectively, for the incomplete elliptic function of the third kind  $\Pi$ .

APPENDIX B. The Maclaurin series expansion of the elliptic integral of the third kind coincides with a triple series [27]

$$\begin{aligned} \Pi(n; \varphi|m) &= \frac{\sqrt{\pi} (m/2)^{3/2}}{n(m - 2(1 + \sqrt{1-m}))^{3/2} \sqrt{1-n}} \sqrt{1 + \frac{1}{\sqrt{1-m}}} \sum_{q=0}^{\infty} \frac{(-4)^q}{(2q+1)!} \sum_{k=0}^{2q} \frac{S_{2q}^{(k)}}{2^k} \\ &\times \sum_{j=0}^k \frac{(-n)^{-j} j! \binom{k}{k-j}}{\Gamma(j - k + \frac{1}{2})} \frac{m^{k-j} ((1 + \sqrt{1-n})^{j+1} - (1 + \sqrt{1-n})^{j+1})}{(1 - \sqrt{1-m})^{k-j}} \\ &\times F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{3}{2}; j - k + \frac{1}{2}; \frac{1}{2} - \frac{1}{2\sqrt{1-m}}, 2 \frac{1 + \sqrt{1-m}}{m}\right) \varphi^{2q+1}, \end{aligned}$$

where the Stirling numbers of the second kind

$$S_b^{(a)} = \frac{(-1)^a}{a!} \sum_{m=0}^a (-1)^m \binom{a}{m} m^b, \quad a, b - 1 \in \mathbb{N}$$

and the Appell function [40, p. 21, Eq. (2.1.1)]

$$F_1(a; b, b'; c; x, y) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} (b)_r (b')_s}{(c)_{r+s} r! s!} x^r y^s, \quad |x| < 1, |y| < 1,$$

playing roles. Here the familiar Pochhammer symbol (or shifted factorial)

$$(a)_n = a(a+1) \cdots (a+n-1) \quad a \in \mathbb{C}, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\},$$

and  $0_0 = 1$  is used by convention. Specifying  $n = m = e^2$ , the previous formulae give (5).

APPENDIX C. The Legendre polynomial  $y = P_n(x)$  is a solution of the Legendre differential equation, reads as follows

$$((1-x^2)y')' + n(n+1)y = 0;$$

the solution may be expressed by the Rodrigues formula [4, p. 252]

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1-x^2)^n.$$

The associated Gauss–Legendre low order ( $n = 1, 2, 3, 4$ ) quadrature weight–coefficients

$$w_j = \frac{2}{(1-t_j^2)[P'_n(t_j)]^2}, \quad j = \overline{1, n}, \quad n = 1, 2, 3, 4$$

are presented on the following table

| $n$ | nodes $t_j$  | weights $w_j$  |
|-----|--|--|
| 1   | 0  | 2  |
| 2   | $\pm \frac{1}{\sqrt{3}}$   | 1  |
| 3   | $\pm \sqrt{\frac{3}{5}}; 0$  | $\frac{5}{9}; \frac{8}{9}$   |
| 4   | $\pm \sqrt{\frac{1}{7} \left( 3 - 2\sqrt{\frac{6}{5}} \right)}; \pm \sqrt{\frac{1}{7} \left( 3 + 2\sqrt{\frac{6}{5}} \right)}$ | $\frac{1}{2} + \frac{\sqrt{30}}{36}; \frac{1}{2} - \frac{\sqrt{30}}{36}$ |

Table 2. LOW–ORDER GAUSS–LEGENDRE QUADRATURE WEIGHT–COEFFICIENTS