CHAOS IN HUMAN SYSTEMS--THE HEINE-BOREL THEOREM

One theorem, based in topological notions, that is critical in the development of higher mathematical analysis and that might have potential for application to human systems at the global scale is the Heine-Borel Theorem. This theorem holds in all integral dimensions; for ease in visualization, it is realized here along the real number line. Extensions of this Theorem and of the applications suggested below to fractal sets appear promising, but are beyond the scope of this material [Arlinghaus, 1985].

Definition 1 [Taylor, p. 483].

"A point set S on the x-axis is said to be bounded if there is some finite interval which contains all of S; that is, if there exist numbers a and b, $a < b$, such that $a \le x \le b$ for all $x \in S$."

Definition 2 [Taylor, p. 491]

"Let S be a point set, and suppose we have a collection of a certain number of open sets such that each point of S belongs to at least one of the open sets. Then we say that S is covered by the collection of open sets."

For example, suppose $S = \{x | 0 < x \le 1\}$ and suppose T is a collection of open sets (open intervals on the x-axis) with

$$T = \{I_n = (1/(2^n), (n+2)/(2^n)) | n \text{ is an integer}\}$$

[Taylor, p. 491]. Figure 1 displays the approximate positions of the open intervals on the set S. As n becomes large, $1/(2^n)$ becomes small, but it never reaches 0; the sets I_1, I_2, I_3, and I_4 cover everything in S to the right of 1/16. The infinite sequence of open intervals is required to cover $0 < x \le 1/16$. This set S is covered by the collection T of open intervals, since every x, $0 < x \leq 1$, lies in I_n for some value of n. The set S is bounded, since $0 \leq x \leq 1$ (as in Definition 1); it is not, however, closed--its complement on the real axis is $(-\infty,0] \cup (1,\infty)$ which is not a union of open intervals.

Suppose we make S closed by adding the point 0; $R = S \cup \{0\}$. Does T, as described above, cover this closed and bounded interval R? Clearly it does not, as 0 lies in no I_n. To cover 0, add an interval such as (-1/10, 1/10). Then the infinite collection $U = T \cup \{(-1/10,1/10)\}$ covers R (Figure 2).

The addition of the point 0 to 'close' S, forcing the addition of (-1/10,1/10) to T in order to cover S, has deeper implications. The closed set R may be covered by a finite number of judiciously selected intervals from U; the intervals I_1, I_2, I_3, and I_4 cover everything in R to the right of 1/16, while the added interval, (-1/10,1/10), covers all of R, including 1/16, to the left of 1/10. Thus, five intervals may be used to replace an infinite collection in covering the set R.

Clearly, this is not the case in the situation shown in Figure 1. For, if the intervals $I_1,\ldots,I_k$ (k an arbitrary positive integer) were considered as a finite candidate-set, the interval $0 < x \leq 1/(2^k)$ would remain uncovered. Thus, no finite subset of the infinite collection of open intervals T will cover $S = \{x \mid 0 < x \leq 1\}$.

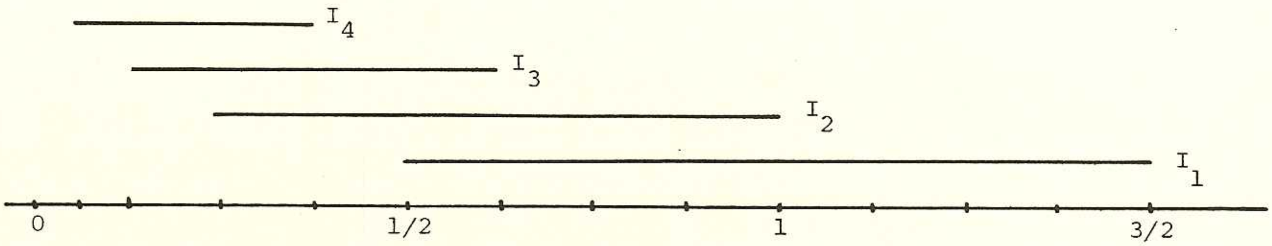
The notion of being able to select a finite subcover from a given covering is the thrust of the Heine-Borel Theorem.

<u>Heine-Borel Theorem</u> [Taylor, p. 493]

"Let S be a bounded and closed point set, and let S be covered by a collection [T] of open sets. Then a finite number of open sets may be chosen from the collection [T] in such a way that S is covered by the new finite collection."

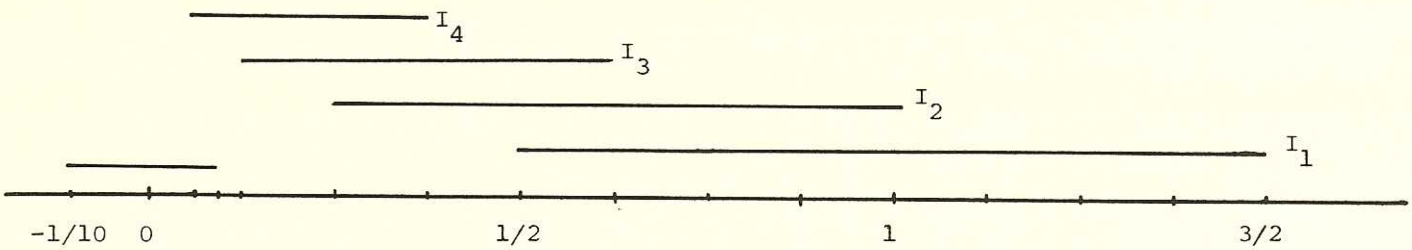
The examples above give the idea of the proof of this theorem; a rigorous proof may be found in Taylor or Rudin.

To apply this Theorem to human systems, suppose that S is a collection of human systems and that T is a collection of interpretations of those systems. Using the Heine-Borel Theorem, we see that if S is closed and bounded, and is covered by T, then from T a finite collection also covering S can be chosen. We might expect S to be bounded by a geographic region (possibly the whole earth), and S might be viewed as closed, if no new input were required from different systems to ensure the functioning of S. The collection T clearly could be finite; however, it rests on belief systems, value systems, and a variety of other social and cultural factors which might be infinite. If the Heine-Borel Theorem holds, a finite number of these views may be chosen that cover, or produce understanding of and rational response to, these systems. In this case, some uniformity in interpretation of the systems is possible in the geographic region containing them. When the number of elements in the finite cover is small, and when the geographic region bounding S is large, global harmony is maximized. Conversely, when the Heine-Borel Theorem does not hold, an infinite collection of interpretations may be required to cover, or understand and respond to, even a fairly small set of human systems (compare to the motivational examples). This suggests cultural chaos; empirical studies along these lines might be drawn from Middle Eastern politics, from English/Irish relations, or from widespread terrorism across the surface of the Earth.



(S,T): $I_1 = (1/2, 3/2)$
$I_2 = (1/4, 4/4)$
$I_3 = (1/8, 5/8)$
$I_4 = (1/16, 6/16)$
....

FIGURE 1



(R,U): $I_1 = (1/2, 3/2)$
$I_2 = (1/4, 4/4)$
$I_3 = (1/8, 5/8)$
$I_4 = (1/16, 6/16)$
....
and $(-1/10, 1/10)$

FIGURE 2

LITERATURE CITED

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3. Taylor, A. E. Advanced Calculus. Boston: Ginn and Co., 1955.